FACULTY OF SCIENCE

MASTER PROGRAM OF MATHEMATICS

USING SYMMETRIES TO SOLVE SOME DIFFERENCE EQUATIONS

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Supervised by

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Declaration

I certify that this Thesis, submitted for the degree of Master of Mathematics to the Department of Mathematics at Birzeit University, is of my own research except where otherwise acknowledged, and that this thesis (or any part of it) has not been submitted for a higher degree to any other university or institution.

Wala'a Yassen	Signature
April, 2018	....

Abstract

We study symmetry method to solve some difference equations by determining Lie groups of symmetries. Then we use these groups to achieve successive reductions of order. If there are enough symmetries, the difference equations can be completely solved.

Keywords: Difference equations; Lie groups; Symmetry method.

الملخص

الهدف من الرسالة هو دراسة طريقة التماثل لحل بعض المعادلات التفاضلية المنفصلة من خلال تحديد مجموعات التماثل. ثم سنستخدم هذه المجموعات لتقليل رتبة المعادلات. إذا كان هناك ما يكفي من التماثلات، سنتمكن من حل المعادلات التفاضلية المنفصلة بشكل كامل.

الكلمات المفتاحية: المعادلات التفاضلية المنفصلة، طريقة التماثل، مجموعات التماثل.

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Symbols

$\mathbb{Z}$	Integer numbers
$\mathbb{N}$	Natural numbers
$\mathbb{R}$	Real numbers
ODE	Ordinary differential equation
PDE	Partial differential equation
$O\Delta E$	Ordinary difference equation
u_n	u(n)
Δ	Forward difference operator
S	Forward shift operator
I	Identity operator
Γ_0	Trivial symmetry
LSC	Linearized symmetry condition
$Q(n, u_n)$	Characteristic of local lie group
X	Infinitesimal generator
s_n	Canonical coordinate
v_n	Invariant

Chapter 1

Introduction

Most methods for solving a given ordinary differential equation ODE use change of variable, which transform the equation into a simpler equation that is easy to solve. This idea was introduced by Sophus Lie. He used symmetry to solve differential equations by determining Lie groups of symmetries of a given ordinary differential equation. For an introduction to symmetry method for ODEs, see [Olver(1993)and Hydon(2000)].

Meada (1987) has shown that difference equations of order *one* can be solved by Lie's method, and he showed that the linearized symmetry condition (LSC) for such difference equation leads to a set of functional equations. Later, Quisple and Sahdevan (1993) were interested in this method and they extended Meada's idea to a higher order difference equations by using a Laurent series expansion about a fixed point at infinity. This method is restricted by the existence of such a fixed point. Levi et al. (1997) expanded the linearized symmetry condition as a series in powers of u_n and looked for symmetries that are more general than point symmetries but the expression derived by them was complicated. Hydon (2000) introduced a method for obtaining the Lie symmetries and used it to reduce the order of the ordinary difference equations and to find the solution. Then, he applied this method to second order difference equations.

In this Thesis, we study the symmetry analysis for ordinary difference equations. We investigate the exact solutions of $second$, $third$ and $fourth$ order nonlinear difference equations using a group of transformations (Lie symmetries).

This Thesis is organized as follows, in chapter two, we introduce some basic concepts and solutions of some types of difference equations. In chapter three, we investigate symmetries of difference equations and the linearized symmetry condition for first and second order difference equations, and we show how can we use it to solve these equations. Finally, we generalize the symmetry method for higher order difference equations.

In chapter four, we apply the symmetry method to solve some nonlinear difference equations.

Notice that, throughout this thesis we will not talk about qualitative theory of difference equations. In particular, there is no discussion of stability or oscillation theory. We introduce knowledge of solution methods for difference equations.

Chapter 2

Basic Preliminaries

2.1 General Basics

In this section, we recall some basic concepts of difference equations.

Definition 2.1.1. *[12] **Difference Equation** is an equation that expresses a value of a sequence as a function of the other terms in the sequence, that is, it defines a relation recursively.*

Definition 2.1.2. *[1] **The order of a difference equation** is the difference between highest and lowest indices that appear in the equation.*

An Ordinary Difference Equation of order p is an equation of the form

$$u(n+p) = F(p, u(n+p-1), \cdots, u(n)), \tag{2.1}$$

where F is a well defined function of it's arguments.

Definition 2.1.3. *[7] A difference equation is linear if equation (2.1) can be written in the form*

$$a_p(n)u_{n+p} + a_{p-1}(n)u_{n+p-1} + \cdots + a_0(n)u_n = b(n), \tag{2.2}$$

where $a_i(n)$ and $b(n)$ for all $i = 0, 1, \cdots, p$ are given functions of n.

Definition 2.1.4. *[7] A difference equation is nonlinear if it is not linear.*

Definition 2.1.5. *[11] A solution of a difference equation is a function $\phi(n)$ that reduces the equation to an identity.*

Linear difference equations can be classified into homogeneous or non-homogeneous equation. That is,

1. If $b(n) \equiv 0$ in equation (2.2) then it's called a homogeneous linear difference equation.

2. If $b(n) \not\equiv 0$ in equation (2.2) then it's called a non-homogeneous linear difference equation.

Now, if the difference equation is nonlinear, then it could be transformed into a linear difference equation, and this property helps us to find a solution. We now give some examples of difference equations.

Example 2.1. *Consider the following difference equations:*

- $3u_{n+2} - u_{n+1} = u_n$. *(2^{nd} order homogeneous linear difference equation).*
- $u_{n+1} = e^{u_n}$. *(1^{st} order nonlinear difference equation).*
- $u_{n+3} - \frac{n}{n+1}u_n = n$. *($3^{rd}$ order non-homogeneous linear difference equation).*

Definition 2.1.6. *[11] An initial value problem of a difference equation is a problem of finding a function that satisfies the equation when we know its value u_0 at a particular point n_0.*

Example 2.2. *The function $\phi(n) = 3^n\left(2 + \frac{n(n-1)}{6}\right)$ is a solution for the initial value problem*

$$u_{n+1} - 3u_n = 3^n n;\ n \geq 0 \quad \textit{and} \quad u_0 = 2,$$

since if we substitute $\phi(n)$ into the equation, we get

$$\begin{aligned} 3^{n+1}\left(2 + \frac{n(n+1)}{6}\right) - 3^{n+1}\left(2 + \frac{n(n-1)}{6}\right) &= 3^{n+1}\left(2 + \frac{n^2}{6} + \frac{n}{6} - 2 - \frac{n^2}{6} + \frac{n}{6}\right) \\ &= 3^n n. \end{aligned}$$

Also, we have

$$\phi(0) = 3^0\left(2 + \frac{0(0-1)}{6}\right) = 2 = u_0.$$

2.2 Existence And Uniqueness Theorem

It should be clear that for a given difference equation, even if a solution is known to exist, there is no assurance that it will be unique. The solution must be restricted by given a set of initial conditions equal in number to the order of the equation. The following theorem states conditions that assure the existence of a unique solution.

Theorem 2.2.1. *[11] Let*

$$u(n+p) = F(n, u(n), \cdots, u(n+p-1)); n = 0, 1, 2, ... \tag{2.3}$$

be a p^{th} order difference equation, where f is defined for each of its arguments. Then equation (2.3) has a unique solution corresponding to each arbitrary selection of the p initial values $u(0), u(1), \cdots, u(p-1)$.

Proof. Suppose that $u(0), u(1), \cdots, u(p-1)$ are given. Then the difference equation with $n = 0$ uniquely specifies $u(p)$. Now $u(p)$ is known, the difference equation with $n = 1$ gives $u(p+1)$. Continue in this way, all u_n for $n \geq p$, can be determined. □

Definition 2.2.1. *[11] The functions $f_1(n), f_2(n), ..., f_m(n)$ are said to be linearly dependent for $n \geq n_0$, if there exists scalars $c_1, c_2, ..., c_m$ not all zero such that*

$$c_1 f_1(n) + c_2 f_2(n) + ... + c_m f_m(n) = 0, \quad \forall n \geq n_0.$$

So each function f_j for $j = 1, 2, \cdots, m$ with nonzero coefficient is a linear combination of the other f_i's. The functions $f_1(n), ..., f_m(n)$ are said to be linearly independent for $n \geq n_0$ if whenever

$$c_1 f_1(n) + c_2 f_2(n) + ... + c_m f_m(n) = 0, \quad \forall n \geq n_0,$$

then we must have $c_1 = c_2 = ... = c_m = 0$.

2.3 First Order Linear Difference Equations

In this section, we consider the simplest linear difference equation which is first order linear difference equation. So we start with the following equation

$$u_{n+1} = a u_n, \quad n \in \mathbb{N} \tag{2.4}$$

where a is a given constant. The solution is given by

$$u_n = a^n u_0. \tag{2.5}$$

The value u_0 is called the initial value. To prove that (2.5) solves (2.4), we proceed as follows:

$$u_{n+1} = a^{n+1} u_0 = a(a^n u_0) = a u_n.$$

Equation (2.4) is a first order homogeneous difference equation with constant coefficients. Now, we want to generalize equation (2.4) to non-homogeneous with non-constant coefficients.

Theorem 2.3.1. *[12] Let $a(n)$ and $b(n)$ be real sequences where $n \in \mathbb{N}$. Then the first order linear difference equation*

$$u_{n+1} + a(n)u_n = b(n), \tag{2.6}$$

with initial condition $u_0 = c$, has a unique solution of the form

$$u_n = \left(\prod_{i=0}^{n-1} -a(i)\right)c + \sum_{i=0}^{n-1}\left(\prod_{j=i+1}^{n-1} -a(j)\right)b(i). \tag{2.7}$$

Proof. First, we must show that (2.7) satisfies the equation (2.6) and the initial condition. We first write the expression for u_{n+1}

$$u_{n+1} = \left(\prod_{i=0}^{n} -a(i)\right)c + \sum_{i=0}^{n}\left(\prod_{j=i+1}^{n} -a(j)\right)b(i).$$

We then rewrite the last summation above as follows,

$$\sum_{i=0}^{n}\left(\prod_{j=i+1}^{n} -a(j)\right)b(i) = \prod_{j=n+1}^{n}\left(-a(j)b(n)\right) + \sum_{i=0}^{n-1}\left(\prod_{j=i+1}^{n} -a(j)\right)b(i)$$

since

$$\prod_{j=n+1}^{n}\left(-a(j)\right) = 1,$$

we get

$$\begin{aligned}\sum_{i=0}^{n}\left(\prod_{j=i+1}^{n} -a(j)\right)b(i) &= b(n) + \sum_{i=0}^{n-1}\left(\prod_{j=i+1}^{n} -a(j)\right)b(i) \\ &= b(n) - a(n)\left[\sum_{i=0}^{n-1}\left(\prod_{j=i+1}^{n-1} -a(j)\right)b(i)\right].\end{aligned}$$

Using this result we obtain,

$$u_{n+1} = -a(n)\left(\prod_{i=0}^{n-1} -a(i)\right)c + b(n) - a(n)\left(\sum_{i=0}^{n-1}\left[\prod_{j=i+1}^{n-1} -a(j)\right]b(i)\right),$$

which implies

$$u_{n+1} = -a(n)u_n + b(n).$$

Thus, we have shown that u_n is a solution. Finally we must prove uniqueness. Assume that we have two solutions u_n and $\hat{u}_n$, both satisfy (2.6) and the initial condition. Now, consider the set $\{n \in \mathbb{N}; u_n \neq \hat{u}_n\}$. Let n_0 be the smallest integer in this set. We must have $n_0 \geq 1$, since $u_0 = \hat{u}_0$. By the definition of n_0 we have $u_{n_0-1} = \hat{u}_{n_0-1}$ and then

$$u_{n_0} = a(n_0 - 1)u_{n_0-1} + b(n_0 - 1) = a(n_0 - 1)\hat{u}_{n_0-1} + b(n_0 - 1) = \hat{u}_{n_0},$$

which is a contradiction. Thus we must have $n_0 = 0$. But $u_0 = \hat{u}_0 = c$ since the two equations satisfy the same initial condition. It follows that the solution is unique. □

Example 2.3. *Consider the difference equation*

$$u_{n+1} = 2u_n + n, \quad u_0 = 5.$$

Solution. Using the general formula (2.7) we get the solution

$$u_n = 5(2)^n + \sum_{i=0}^{n-1} i(2)^{n-1-i} = 5(2)^n + 2^n - n - 1.$$

■

2.4 Difference Calculus

In this section, we want to define operators which act on difference equations.

Definition 2.4.1. *[7] The forward difference operator Δ is defined as follows*

$$\Delta u_n = u_{n+1} - u_n,$$

where the expression $u_{n+1} - u_n$ is called the difference of u_n. Similarly, we call $\Delta^2 = \Delta.\Delta$ the second difference operator and when acting on u_n, we get

$$\begin{aligned}\Delta^2 u_n =& \Delta(\Delta u_n)\\ =& \Delta(u_{n+1} - u_n)\\ =& u_{n+2} - 2u_{n+1} + u_n.\end{aligned}$$

In general, for any positive integer m, we define the relation

$$\Delta^m u_n = \Delta^{m-1}(\Delta u_n),$$

repeating this $m-$ times.

Any ordinary difference equation can be written in terms of the forward shift operator S and the identity operator I, which are defined as follows

$$S : n \to n+1, \qquad I : n \to n, \qquad \forall n \in \mathbb{Z}. \tag{2.8}$$

The identity operator I maps each function of n to itself. The operator S maps each function of n to a function of $n+1$.
The forward difference operator Δ can be written in terms of the operators S and I

$$\Delta = S - I.$$

If we apply S to any function of n repeatedly by r times, we obtain

$$S^r\{f(n)\} = f(n+r), \qquad S^r u_n = u_{n+r}.$$

The forward shift operator satisfies a simple product rule

$$S^r\{f(n)g(n)\} = f(n+r)g(n+r) = S^r\{f(n)\}S^r\{g(n)\}.$$

Example 2.4. *Any first order linear homogeneous difference equation can be written in operator notation as*

$$(S + a(n)I)u_n = 0.$$

2.5 Higher Order Linear Difference Equations

In this section, we give a short introduction to the theory of higher order linear difference equations. A linear difference equation of order p has the following form

$$a_p(n)u_{n+p} + a_{p-1}(n)u_{n+p-1} + \cdots + a_0(n)u_n = b(n), \tag{2.9}$$

where $a_p(n)$ and $a_0(n)$ are not zeros. As we mentioned before in section (2.1), if $b(n)$ is identically *zero*, then the linear equation is homogeneous and has the form

$$a_p(n)u_{n+p} + a_{p-1}(n)u_{n+p-1} + \cdots + a_0(n)u_n = 0. \tag{2.10}$$

Lemma 2.5.1. *[12] Let $u_1(n)$ and $u_2(n)$ be two solutions of equation (2.10). Then the following statements hold*

1. *$u_n = u_1(n) + u_2(n)$ is a solution of equation(2.10).*

2. $\hat{u}(n) = au_1(n)$; *a is a constant is also a solution of equation (2.10).*

Proof. 1. Let $u_1(n)$ and $u_2(n)$ be two solutions of equation (2.10). So

$$a_p(n)u_1(n+p) + a_{p-1}(n)u_1(n+p-1) + \cdots + a_0(n)u_1(n) = 0$$

and

$$a_p(n)u_2(n+p) + a_{p-1}(n)u_2(n+p-1) + \cdots + a_0(n)u_2(n) = 0.$$

Add the last two equations to each other, we get

$$a_p(n)u(n+p) + a_{p-1}u(n+p-1) + \cdots + a_0(n)u(n) = 0,$$

where $u(n) = u_1(n) + u_2(n)$. So u_n is a solution of equation (2.10).

2. Assume $u_1(n)$ is a solution of equation (2.10), then

$$a_p(n)u_1(n+p) + a_{p-1}(n)u_1(n+p-1) + \cdots + a_0(n)u_1(n) = 0.$$

Now, we multiply the last equation by a this implies that $\hat{u}(n)$ is a solution of equation (2.10).

□

Theorem 2.5.2. *[12] (Superposition Principle) If $u_1(n), u_2(n), \cdots, u_m(n)$ are solutions of equation (2.10), then $u(n) = c_1u_1(n) + c_2u_2(n) + ... + c_mu_m(n)$ is also a solution.*

Proof. Direct from previous Lemma (2.5.1). □

Definition 2.5.1. *[12] A set of m linearly independent solutions of equation (2.10) is called a fundamental set of solutions.*

Definition 2.5.2. *[12] Let $\{u_1(n), u_2(n), \cdots, u_m(n)\}$ be a fundamental set of solutions of equation (2.10). Then the general solution of equation (2.10) is given by*

$$\sum_{i=1}^{m} c_i u_i(n).$$

Now, our objective is to find a fundamental set of solutions and, consequently, the general solution of equation (2.10). First, we want to consider the case where the a_i's are constants and $a_0 \neq 0$, that is, equation (2.10) is simplified to

$$a_p u_{n+p} + a_{p-1}u_{n+p-1} + \cdots + a_0 u_n = 0. \tag{2.11}$$

We suppose that solutions of equation (2.11) are of the form r^n where $n \in \mathbb{N}$. Substituting r^n into equation (2.11), we get

$$a_p r^p + a_{p-1} r^{p-1} + ... + a_0 = 0.$$

This equation is called the characteristic equation of equation (2.11) and its roots $r_1, r_2, \cdots, r_p$ are called the characteristic roots. We have three cases

- Suppose the roots $r_1, r_2, \cdots, r_p$ are distinct and real, then the set $\{r_1^n, r_2^n, ..., r_p^n\}$ is a fundamental set of solutions and the general solution is given by

  $$u_n = \sum_{i=1}^{p} c_i r_i^n, \tag{2.12}$$

  where $c_1, c_2, ..., c_p \in \mathbb{R}$.

- If the roots are distinct complex roots then the general solution could be written in the form (2.12), which can be written in polar form

  $$r_j = \rho_j e^{i\theta_j},$$

  but the complex roots appear in pairs, i.e, if r_j is a root then $\overline{r}_j$ is also a root. So the general solution is

  $$u_n = \sum_{j=1}^{m} r_j^n [c_j \cos(n\theta_j) + \hat{c}_j \sin(n\theta_j)].$$

- Suppose that the characteristic roots $r_1, r_2, ..., r_k$ are distinct with multiplicities $m_1, m_2, ..., m_k$, respectively, such that $\sum_{i=1}^{k} m_i = p$, then the general solution is

  $$\sum_{i=1}^{k} r_i^n (c_{i0} + c_{i1} n + ... + c_{im_{i-1}} n^{m_{i-1}})$$

  where c_{ij}'s $\in \mathbb{R}$.

Example 2.5. *Write the general solutions of the following difference equations:*

1. $u_{n+3} - 7u_{n+2} + 16u_{n+1} - 12u_n = 0.$

   Solution. The characteristic equation is

   $$r^{n+3} - 7r^{n+2} + 16r^{n+1} - 12r^n = 0,$$

which implies that

$$r^3 - 7r^2 + 16r - 12 = 0.$$

So the characteristic roots are $r_1 = 3$ and $r_2 = r_3 = 2$ and the general solution is

$$u_n = c_1 3^n + c_2 2^n + c_3 n 2^n.$$

■

2. $u_{n+2} + 16u_n = 0.$

Solution. The characteristic equation is

$$r^{n+2} + 16r^n = 0,$$

which implies that

$$r^2 + 16 = 0.$$

So the characteristic roots are $r = 4i$ and $-4i$ and the general solution is

$$u_n = 4^n \left(c_1 \cos\left(\frac{n\pi}{2}\right) + c_2 \sin\left(\frac{n\pi}{2}\right)\right).$$

■

Now, we want to focus our attention on solving the p^{th} order linear non-homogeneous equation

$$a_p(n)u_{n+p} + a_{p-1}(n)u_{n+p-1} + \cdots + a_0(n)u_n = b(n), \tag{2.13}$$

where $a_0(n) \neq 0$ and $a_p(n) \neq 0$ for all $n \geq n_0$. The sequence $b(n)$ is called the forcing or external term. This equation represent a physical system in which $b(n)$ is the input and u_n is the output.

Theorem 2.5.3. *[12] If $u_1(n)$ and $u_2(n)$ are solutions of equation (2.13), then $u_n = u_1(n) - u_2(n)$ is a solution of the corresponding homogeneous equation of (2.13).*

Proof. Suppose $u_1(n)$ and $u_2(n)$ are two solutions of equation (2.13), so

$$a_p(n)u_1(n+p) + a_{p-1}u_1(n+p-1) + \cdots + a_0 u_1(n) = b(n),$$

and

$$a_p(n)u_2(n+p) + a_{p-1}(n)u_2(n+p-1) + \cdots + a_0 u_2(n) = b(n).$$

Now, subtract the last two equations, then we get

$$a_p(n)\Bigg(u_2(n+p)-u_1(n+p)\Bigg)+a_{p-1}(n)\Bigg(u_2(n+p-1)-u_1(n+p-1)\Bigg)+\\ \cdots+a_0\Bigg(u_2(n)-u_1(n)\Bigg)=0.$$

So $u_2(n)-u_1(n)$ is a solution of the corresponding homogeneous equation. □

Theorem 2.5.4. *[12] Any solution u_n of equation (2.13) can be written as*

$$u_n = u_p(n) + u_h(n);$$

where u_h is the general solution of the corresponding homogeneous equation, and u_p is a particular solution of the non-homogeneous equation.

Proof. Suppose u_n and $u_p(n)$ are two solutions of equation (2.13), then by theorem (2.5.3), $u_n - u_p(n)$ is a solution of the corresponding homogeneous equation, so

$$u_n - u_p(n) = u_h(n).$$

This implies $u_n = u_p(n) + u_h(n)$. □

As a consequence to theorem (2.5.4), we are left with the problem of finding a particular solution to a given non-homogeneous equation (2.13). First, we want to consider the case where the coefficients a_i's are constant and $b(n)$ is a linear combination or products of the functions

$$k^n, \sin(bn), \cos(bn), \text{ or } n^p.$$

For this case we use the method of Undetermined coefficients to compute $u_p(n)$.
We can summarize this method by the following three steps:

- Solve the corresponding homogeneous equation.
- Verify that $b(n)$ is a linear combination of the functions in the Table (2.1). If $b(n)$ isn't in a form in Table (2.1), then the method can't be applied.
- To determine the coefficients of the particular solution, we substitute the form of the solution in the non-homogeneous equation.

Example 2.6. *Solve the difference equations*

TABLE 2.1: Particular Solutions $u_p(n)$.

$b(n)$	$u_p(n)$
k^n	ck^n
n^p	$c_0 + c_1 n + ... + c_p n^p$
$n^p k^n$	$k^n(c_0 + c_1 n + ... + c_p n^p)$
$\sin(an)$, *or* $\cos(an)$	$c_1 \sin(an) + c_2 \cos(an)$
$k^n \sin(an)$, *or* $k^n \cos(an)$	$k^n(c_1 \sin(an) + c_2 \cos(an))$

1. $u_{n+2} - u_{n+1} - 6u_n = 36n.$

Solution. The characteristic equation is

$$r^{n+2} - r^{n+1} - 6r^n = 0,$$

which implies that

$$r^2 - r - 6 = 0.$$

So the characteristic roots are $r = 3$ and -2 and the solution of the homogeneous equation is

$$u_n = c_1 3^n + c_2(-2)^n.$$

Now, to find the particular solution, let

$$u_p(n) = an + b,$$

substitute in the equation, we get

$$a(n+2) + b - a(n+1) - b - 6a(n) - 6b = 36n,$$

this implies that

$$an + 2a + b - an - a - b - 6an - 6b = 36n,$$

so $a = -6$ and $b = -1$. The general solution is

$$u_n = c_1 3^n + c_2(-2)^n - 6n - 1.$$

■

2. $u_{n+2} + 4u_n = 2^n \sin(\frac{n\pi}{2}).$

Solution. The characteristic equation is

$$r^{n+2} + 4r^n = 0,$$

which implies that

$$r^2 + 4 = 0.$$

So the characteristic roots are $r = 2i$ and $r = -2i$, so the solution of the homogeneous equation is

$$u_n = 2^n \left(c_1 \sin\left(\frac{n\pi}{2}\right) + c_2 \cos\left(\frac{n\pi}{2}\right)\right).$$

In this case, $u_h(n)$ and $b(n)$ are linearly dependent and $b(n)$ is a linear combination of the form of functions in Table(2.1), so the particular solution from Table (2.1) is multiplied by n and it is of the form

$$u_p(n) = n2^n \left(\hat{c}_1 \sin\left(\frac{n\pi}{2}\right) + \hat{c}_2 \cos\left(\frac{n\pi}{2}\right)\right),$$

substitute it in the non-homogeneous equation, we get

$$u_p(n) = \frac{-n}{4} \sin\left(\frac{n\pi}{2}\right).$$

So the general equation is

$$u_n = 2^n \left(c_1 \sin\left(\frac{n\pi}{2}\right) + c_2 \cos\left(\frac{n\pi}{2}\right)\right) - \frac{n}{4} \sin\left(\frac{n\pi}{2}\right).$$

■

But if we look at the general non-homogeneous linear difference equation, we have no general method for solving them. Sometimes, we can guess one solution to this equation, then use the reduction of order method to find a second linearly independent solution. Also, there are other methods for linear difference equations which aren't included in this section as the z-transform.

2.6 Nonlinear Difference Equations

In a linear difference equations, every term of the equation contains at most one of the elements of the sequence $\{u_n\}$, and the elements occur only "as themselves", they

are not raised to any power (other than one). In a nonlinear difference equation, all these restrictions are lifted. Methods of solution for the two different types of equations are very different and the solutions exhibit very different properties. Over a century ago, there was no standard method for finding analytic solutions to nonlinear difference equations. A simple technique could be used to obtain a great deal of information about nonlinear difference equations is to use a fixed-point analysis. The idea is to find particular points for which the solution is fixed, which are not included in this work. In the next Chapter, we introduce a method for nonlinear difference equations, using a method that is very important in solving nonlinear differential equations. This method was developed by Sophus Lie by the end of the 19th century. Meada (1987) has shown that ordinary difference equations can be simplified using Lie's idea.
In this section, we focus on the nonlinear equations which can be transformed into linear equations.

- **Type one. Ricatti Equations:** Difference equations has the form

$$u_{n+1}u_n + a(n)u_{n+1} + b(n)u_n = g(n). \tag{2.14}$$

To solve equation (2.14), we consider the following two cases

1. If $g(n) \equiv 0$, then we let
$$x_n = \frac{1}{u_n},$$
substitute in equation (2.14), we obtain
$$\frac{1}{x_{n+1}}\frac{1}{x_n} + a(n)\frac{1}{x_{n+1}} + b(n)\frac{1}{x_n} = 0,$$
then multiply by $x_{n+1}x_n$ to get
$$b(n)x_{n+1} + a(n)x_n + 1 = 0;$$
which is linear difference equation.

2. If $g(n) \not\equiv 0$, then we let
$$u_n = \frac{x_{n+1}}{x_n} - a(n). \tag{2.15}$$

Now, substitute (2.15) into (2.14)

$$\begin{aligned} u_{n+1}u_n + a(n)u_{n+1} + b(n)u_n &= u_{n+1}(u_n + a(n)) + b(n)u_n \\ &= \left(\frac{x_{n+2}}{x_{n+1}} - a(n+1)\right)\left(\frac{x_{n+1}}{x_n}\right) + b(n)\left(\frac{x_{n+1}}{x_n} - a(n)\right) \\ &= \frac{x_{n+2}}{x_n} - \frac{a(n+1)x_{n+1}}{x_n} + \frac{b(n)x_{n+1}}{x_n} - a(n)b(n) \\ &= g(n), \end{aligned}$$

that is,

$$\frac{x_{n+2}}{x_n} - \frac{a(n+1)x_{n+1}}{x_n} + \frac{b(n)x_{n+1}}{x_n} - a(n)b(n) = g(n),$$

multiply by x_n, we get

$$x_{n+2} + (b(n) - a(n+1))x_{n+1} - (g(n) + a(n)b(n))x_n = 0,$$

which is linear difference equation.

- **Type 2. Equations of general Riccati type:**

$$u_{n+1} = \frac{a(n)u_n + b(n)}{c(n)u_n + d(n)}, \tag{2.16}$$

where $c(n) \neq 0$, and $a(n)d(n) - b(n)c(n) \neq 0$ for all $n \geq 0$.
To solve it, we let

$$c(n)u_n + d(n) = \frac{z_{n+1}}{z_n},$$

then we substitute

$$u_n = \frac{z_{n+1}}{c(n)z_n} - \frac{d(n)}{c(n)},$$

into equation (2.16), we obtain

$$\left(\frac{z_{n+2}}{c(n+1)z_{n+1}} - \frac{d(n+1)}{c(n+1)}\right)\left(\frac{z_{n+1}}{z_n}\right) = a(n)\left(\frac{z_{n+1}}{c(n)z_n} - \frac{d(n)}{c(n)}\right) + b(n).$$

Multiply this equation by $c(n+1)z_n$, we get

$$z_{n+2} - d(n+1)z_{n+1} - a(n)\frac{c(n+1)z_{n+1}}{c(n)} + \left(\frac{a(n)d(n)c(n+1)}{c(n)} - b(n)c(n+1)\right)z_n = 0,$$

which is equivalent to

$$z_{n+2} - \left(d(n+1) - a(n)\frac{c(n+1)}{c(n)}\right)z_{n+1} + \left(\frac{a(n)d(n)c(n+1)}{c(n)} - b(n)c(n+1)\right)z_n = 0,$$

this equation is of the form

$$z_{n+2} + g_1(n)z_{n+1} + g_2(n)z_n = 0,$$

which is linear difference equation.

Example 2.7. *Solve the difference equation*

$$u_{n+1} = \frac{2u_n + 4}{u_n - 1}.$$

Solution. Here $a = 2$, $b = 4$, $c = 1$, and $d = -1$. Hence

$$ad - bc = 2(-1) - 4(1) = -6 \neq 0,$$

so we let

$$u_n - 1 = \frac{z_{n+1}}{z_n}, \tag{2.17}$$

we obtain

$$z_{n+2} - z_{n+1} - 6z_n = 0.$$

The characteristic equation is

$$r^{n+2} - r^{n+1} - 6r^n = 0,$$

which implies that

$$r^2 - r - 6 = 0,$$

so the characteristic roots are: $r = 3$ and $r = -2$ and the general solution is

$$z_n = c_1 3^n + c_2(-2)^n,$$

where c_1 and $c_2 \in \mathbb{R}$. From (2.17) we have

$$\begin{aligned} u_n &= \frac{c_1(3)^{n+1} + c_2(-2)^{n+1}}{c_1(3)^n + c_2(-2)^n} + 1 \\ &= \frac{4c_1(3)^n - c_2(-2)^n}{c_1(3)^n + c_2(-2)^n}, \end{aligned}$$

where c_1 and $c_2 \in \mathbb{R}$. ■

- **Type 3. Homogeneous Difference Equations:** Homogeneous Difference Equations are equations of the form

$$g\left(\frac{u_{n+1}}{u_n}, n\right) = 0, \quad \text{where } u_n \neq 0.$$

To solve difference equations of this form, we let

$$z_n = \frac{u_{n+1}}{u_n},$$

after this substitution we get a difference equation which is linear in z_n.

Example 2.8. *Solve the difference equation*

$$u_{n+1}^2 - 2u_{n+1}u_n - 3u_n^2 = 0. \tag{2.18}$$

Solution. Multiplying equation (2.18) by $\frac{1}{u_n^2}$, we obtain

$$\frac{u_{n+1}^2}{u_n^2} - 2\frac{u_{n+1}}{u_n} - 3 = 0,$$

so we let

$$z_n = \frac{u_{n+1}}{u_n},$$

we get

$$z_n^2 - 2z_n - 3 = 0,$$

so $z_n = 3$ or $z_n = -1$, which implies

$$u_{n+1} = 3u_n \text{ or } \; u_{n+1} = -u_n,$$

which are linear difference equations, whose solutions are

$$u_n = c_1 3^n \text{ or } \; u_n = c_2(-1)^n,$$

where $c_1, c_2 \in \mathbb{R}$. ■

- **Type 4. Consider the difference equation:**

$$u_{n+p}^{k_1} u_{n+p-1}^{k_2} \ldots u_n^{k_{p+1}} = g(n).$$

To solve this equation, we let

$$z_n = \ln u_n,$$

which implies

$$k_1 z_{n+p} + k_2 z_{n+p-1} + ... + k_{p+1} z_n = \ln g(n),$$

which is linear in z_n.

Example 2.9. *Solve*

$$u_{n+2} = \frac{u_{n+1}^3}{u_n^2}.$$

Solution. Let

$$z_n = \ln u_n,$$

then we obtain

$$z_{n+2} - 3z_{n+1} + 2z_n = 0.$$

The characteristic equation is:

$$r^{n+2} - 3r^{n+1} + 2r^n = 0,$$

which implies that

$$r^2 - 3r + 2 = 0,$$

so the characteristic roots are $r = 2$ and $r = 1$. The general solution is

$$z_n = c_1 2^n + c_2,$$

where c_1 and $c_2 \in \mathbb{R}$. Thus,

$$u_n = \exp(c_1 2^n + c_2).$$

■

2.7 Taylor Series

Definition 2.7.1. *[3] If $f(x)$ is a function which is infinitely differentiable at a, the Taylor Series of the function $f(x)$ at/about a is the power series*

$$\begin{aligned} T(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots . \end{aligned}$$

If $a = 0$, then this series is called the Maclaurin Series of the function f given by

$$\begin{aligned} T(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &= f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots . \end{aligned}$$

If $T(x)$ is defined in an open interval around a, then it is differentiable at a, since it is a power series. Furthermore, every derivative of $T(x)$ at a equals the corresponding derivative of $f(x)$ at a.

Theorem 2.7.1. *[3](Taylor's Formula with Remainder) Let f be a function whose (n + 1)th derivative $f^{(n+1)}(x)$ exists for each x in an open interval I containing a. Then, for each $x \in I$,*

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x),$$

where the remainder $R_n(x)$ is given by

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

for some c between x and a.

Example 2.10. *Find the Taylor Series for $\ln x$ about $x = 1$.*

Solution. Calculating the derivatives of $\ln x$ and evaluating them at $x = 1$ gives

$$\begin{aligned} f(x) = \ln x &\quad\Rightarrow\quad f(1) = 0, \\ f'(x) = \frac{1}{x} &\quad\Rightarrow\quad f'(1) = 1, \\ f''(x) = \frac{-1}{x^2} &\quad\Rightarrow\quad f''(1) = -1, \\ f'''(x) = \frac{2}{x^3} &\quad\Rightarrow\quad f'''(1) = 2, \end{aligned}$$

from this, we obtain the pattern that

$$f^{(n)}(1) = (-1)^{n+1}(n+1)!$$

It follows that the Taylor Series for $\ln x$, centered at $a = 1$, is

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots .$$

■

2.8 Method Of Characteristics

In this section, we describe a general technique for solving a special first-order partial differential equations.

A first order partial differential equation is quasi linear if it is linear in the derivatives of the dependent variables. Each term is a product of a function $f(x, y, u)$ and 1 or derivatives of u. That is, each linear first order partial differential equation is quasi linear but the converse isn't true.

Example 2.11. *Examples of quasi linear partial differential equations*

- $u_x + u_y + u^2 = 0.$
- $u_x + u^3 u_y = 5xy.$
- $u_x + 3x^3 y u_y = 2u.$

Any first order quasi linear PDE can be written as

$$a(x, y, z)z_x + b(x, y, z)z_y = c(x, y, z), \tag{2.19}$$

Such equations occur in a variety of nonlinear wave propagation problems. Let us assume that an integral surface $z = z(x, y)$ of equation (2.19) can be found. Writing this integral surface in implicit form as

$$F(x, y, z) = z(x, y) - z = 0.$$

Note that the gradient vector $\nabla F = \langle z_x, z_y, -1\rangle$ is normal to the integral surface $F(x, y, z) = 0$. The equation (2.19) may be written as

$$az_x + bz_y - c = \langle a, b, c\rangle \cdot \langle z_x, z_y, -1\rangle = 0. \tag{2.20}$$

This shows that the vector $\langle a, b, c\rangle$ and the gradient vector ∇F are orthogonal. In other words, the vector $\langle a, b, c\rangle$ lies in the tangent plane of the integral surface $z = z(x, y)$ at each point in the (x, y, z)-space where $\nabla F \neq 0$. At each point (x, y, z), the vector $\langle a, b, c\rangle$ determines a direction in (x, y, z)-space is called the characteristic direction. We can construct a family of curves that have the characteristic direction at each point. If the parametric form of these curves is

$$x = x(t), \;\; y = y(t), \;\; \text{and} \;\; z = z(t), \tag{2.21}$$

then we must have

$$\frac{dx}{dt} = a(x(t), y(t), z(t)), \frac{dy}{dt} = b(x(t), y(t), z(t)), \frac{dz}{dt} = c(x(t), y(t), z(t)), \tag{2.22}$$

because $\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle$ is the tangent vector along the curves. The solutions of (2.22) are called the characteristic curves of the quasi linear equation (2.19). We assume that $a(x, y, z)$, $b(x, y, z)$, and $c(x, y, z)$ are sufficiently smooth and do not all vanish at the same point. Then, the theory of ordinary differential equations ensures that a unique characteristic curve passes through each point (x_0, y_0, z_0). The initial value problem (IVP) for equation (2.19) requires that $z(x, y)$ be specified on a given curve in (x, y)-space which determines a curve C in (x, y, z)-space referred to as the initial curve. To solve this IVP, we pass a characteristic curve through each point of the initial curve C. If these curves generate a surface known as integral surface. This integral surface is the solution of the IVP.

Remark 1. *The characteristics equations (2.22) can be expressed in the nonparametric form as*

$$\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{c}. \tag{2.23}$$

Below, we shall describe a method for finding the general solution of equation (2.19). This method is due to Lagrange. Hence, it is usually referred to as the method of characteristics or the method of Lagrange.

The method of characteristics

It is a method of solution of quasi linear PDE which is stated in the following result.

Theorem 2.8.1. *[2] The general solution of the quasi linear PDE (2.19) is*

$$F(u, v) = 0, \tag{2.24}$$

where F is an arbitrary function and $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ form a solution of the equations (2.23).

Proof. If $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ satisfy the equation (2.19) then the equations

$$u_x dx + u_y dy + u_z dz = 0,$$

$$v_x dx + v_y dy + v_z dz = 0,$$

are compatible with equation (2.23). Thus, we must have

$$au_x + bu_y + cu_z = 0,$$

$$av_x + bv_y + cv_z = 0.$$

Solving these equations for a, b and c, we obtain

$$\frac{a}{\frac{\partial(u,v)}{\partial(y,z)}} = \frac{b}{\frac{\partial(u,v)}{\partial(z,x)}} = \frac{c}{\frac{\partial(u,v)}{\partial(x,y)}}. \tag{2.25}$$

Differentiate $F(u, v) = 0$ with respect to x and y, respectively, to have

$$\frac{\partial F}{\partial u}\left\{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right\} + \frac{\partial F}{\partial v}\left\{\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right\} = 0,$$

and

$$\frac{\partial F}{\partial u}\left\{\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right\} + \frac{\partial F}{\partial v}\left\{\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right\} = 0.$$

Eliminating $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ from these equations, we obtain

$$\frac{\partial z}{\partial x}\frac{\partial(u,v)}{\partial(y,z)} + \frac{\partial z}{\partial y}\frac{\partial(u,v)}{\partial(z,x)} = \frac{\partial(u,v)}{\partial(x,y)}. \tag{2.26}$$

In view of (2.25), the equation (2.26) yields

$$a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = c.$$

Thus, we find that $F(u, v) = 0$ is a solution of the equation (2.19). This completes the proof. □

Example 2.12. *Find the general solution of*

$$xz_x + yz_y = z.$$

Solution. The associated system of equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}.$$

From the first two relation we let first

$$\frac{dx}{x} = \frac{dy}{y},$$

we get

$$\ln x = \ln y + \ln c_1,$$

this implies

$$\frac{x}{y} = c_1.$$

Similarly, we let

$$\frac{dz}{z} = \frac{dy}{y},$$

we get

$$\frac{z}{y} = c_2,$$

where c_1 and c_2 are arbitrary constants. Thus, the general solution is the general integral given by

$$F\left(\frac{x}{y}, \frac{z}{y}\right) = 0,$$

where F is an arbitrary function. ■

Chapter 3

Symmetry Method

Symmetry is a universal concept in nature, science, and art. A symmetry of a geometrical object is an invertible transformation whose action specifies the object to itself. The points themselves may be changed, but the whole object stays as it is. For example, consider the rotation of a regular Hexagon about its diameters ab or cd or ef (see Figure 3.1). The Hexagon is mapped to itself if the angle of rotation is an integer multiple of $\frac{\pi}{3}$, so this transformation is a symmetry.

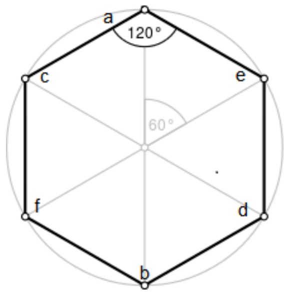

FIGURE 3.1: Symmetries Of A Hexagon

Definition 3.0.1. ***Transformation*** *or a mapping of a region R_1 into a region R_2 is a rule that assign's to each point $p \in R_1$ a unique point $q \in R_2$.*

Definition 3.0.2. *[9]* ***Trivial Symmetry*** *is the transformation that maps each point to itself.*

Remark 2. *From the Definition (3.0.2), every object has at least one symmetry, which is the trivial symmetry.*

Any symmetry must preserves the shape of the object, that is the distance between any two points of the object must be preserved, consequently, the only transformations of Euclidean space are consisting of rotations, translations, and reflections.
So in summary, we can define symmetry in the following definition.

Definition 3.0.3. *[9] A transformation is a symmetry if it satisfies the following properties:*

1. *The transformation preserves the structure.*
2. *The transformation is a diffeomorphism, that is a smooth invertible mapping whose inverse is also smooth.*
3. *The transformation maps the object to itself.*

Definition 3.0.4. *[10] A group is a set G together with a group operation (usually called multiplication) such that for any two elements g and h of G, the product $g \cdot h$ is again an element of G. The group operator is required to satisfy the following axioms:*

- *Associativity. If g, h and k are elements of G, then*
$$(g \cdot h) \cdot k = g \cdot (h \cdot k).$$
- *Identity element. There is a distinguished element e of G, called the identity element, which has the property that*
$$e \cdot g = g \cdot e = g$$
for all $g \in G$.
- *Inverses.For each g in G there is an inverse, denoted g^{-1} with the property*
$$g \cdot g^{-1} = g^{-1} \cdot g = e$$
.

Theorem 3.0.1. *[7] Let G be the set of all symmetries of a geometrical object, then G is a group.*

Proof. Let Γ_a and Γ_b be two symmetries of an object. Then the composite transformations $\Gamma_a\Gamma_b$, and $\Gamma_b\Gamma_a$ are symmetries of this object, because they are invertible and they keep the object unchanged.
From Remark (2) the trivial symmetry denoted by Γ_0 is the identity map, that is, for any $\Gamma_a \in G$,
$$\Gamma_a\Gamma_0 = \Gamma_0\Gamma_a = \Gamma_a.$$

Furthermore, for any $\Gamma_a \in G$, the transformation that reverts the object to its original state, is the inverse of a transformation, that is,

$$\Gamma_a \Gamma_a^{-1} = \Gamma_a^{-1} \Gamma_a = \Gamma_0.$$

It's clear that, composition of transformations is associative, so G is group. $\square$

Remark 3. *If Γ_a and Γ_b are two symmetries of an object with the property that $\Gamma_a \Gamma_b = \Gamma_b \Gamma_a$, then G is abelian.*

Example 3.1. *The symmetries of the Euclidean real line $\mathbb{R}$ include every translation:*

$$\Gamma_a : x \to x + a,$$

where a is a fixed real number. We note that Γ_a is a symmetry for all $a \in \mathbb{R}$.

3.1 Symmetries Of Difference Equations

A transformation of a difference equation is a symmetry if every solution of the transformed equation is a solution of the original equation and vice verse.

Example 3.2. *Let*

$$\Gamma_a : u_n \to \hat{u}_n = a u_n, \quad \forall\ a \in \mathbb{R} - \{0\},$$

be a transformation on a linear homogeneous difference equation of order p:

$$a_p(n) u_{n+p} + a_{p-1}(n) u_{n+p-1} + \cdots + a_0(n) u_n = 0.$$

Then Γ_a is a symmetry of the difference equation for all $a \in \mathbb{R} - \{0\}$, since if $U_1(n), U_2(n), \cdots, U_p(n)$ are linearly independent solutions, then the general solution is

$$u_n = \sum_{i=1}^{p} c_i U_i(n).$$

The transformation Γ_a maps this solution to

$$\hat{u}_n = a \sum_{i=1}^{p} c_i U_i(n) = \sum_{i=1}^{p} \hat{c}_i U_i(n), \quad \textit{where} \quad \hat{c}_i = a c_i,$$

for all $i = 1, 2, \cdots, p$. So $\hat{u}_n$ is a solution of the original equation and vice verse. Thus, Γ_a is a symmetry for all $a \in \mathbb{R} - \{0\}$.

Consider the set of transformations $G = \{\Gamma_a : \ a \in \mathbb{R} - \{0\}\}$. Then G is a group with the composition $\Gamma_a \Gamma_b = \Gamma_{ab}$, for all $a,\ b \in \mathbb{R}$. Note that, Γ_1 is the identity map and

$\Gamma_a^{-1} = \Gamma_{a^{-1}} = \Gamma_{\frac{1}{a}}$. Furthermore, $\hat{u}_n$ is an analytic function of the parameter a and each element Γ_a in G has the property of a near identity map for all a sufficiently near 1.

Definition 3.1.1. *[6] Consider the following point transformation*

$$\Gamma_a : x \to \hat{x}(x; a), \qquad a \in (a_0, a_1),$$

where $a_0 < 0$ and $a_1 > 0$. Then Γ_a is one parameter local Lie group if the following conditions are satisfied

1. *Γ_0 is the identity map, that is, $\hat{x} = x$ when $a = 0$.*
2. *$\Gamma_a \Gamma_b = \Gamma_{a+b}$, $\forall \ \ a, \ b$ sufficiently close to 0.*
3. *Each $\hat{x}$ can be represented by a Taylor series in a, so*

$$\hat{x}(x; a) = x + a\xi(x) + O(a^2).$$

The term 'point' is used because $\hat{x}$ depends only on the point x.
From conditions 1 and 2 we have $\Gamma_a^{-1} = \Gamma_{-a}$ when $\mid a \mid$ is sufficiently small. A local Lie group may not be a group, except if it satisfies the $four$ group axioms.
In general, a one parameter local Lie group of symmetries of a difference equation will depend on n and u_n.
For simplicity, we call symmetries that belong to a one parameter local Lie group, Lie symmetries.

Example 3.3. *[7] Consider the difference equation:*

$$u_{n+1} - u_n = 0. \tag{3.1}$$

and the transformation

$$\Gamma_\epsilon : (n, u_n) \to (\hat{n}, \hat{u}_n) = (n, u_n + \epsilon); \quad \epsilon \in \mathbb{R} \tag{3.2}$$

Γ_ϵ is a one parameter local Lie group, since

$$\Gamma_0 : (n, u_n) \to (\hat{n}, \hat{u}_n) = (n, u_n),$$

so Γ_0 is the identity map and

$$\Gamma_\delta : (n, u_n) \to (n, u_n + \delta),$$

which implies that

$$\Gamma_\epsilon \Gamma_\delta : (n, u_n + \delta) \to (n, u_n + \delta + \epsilon).$$

Thus,

$$\Gamma_\epsilon \Gamma_\delta = \Gamma_{\epsilon+\delta}.$$

Moreover, each $\hat{u}_n$ can be represented as a Taylor series in ϵ, and Γ_ϵ is a symmetry for equation (3.1) since the solution of (3.1) is $u_n = c$, and every transformation with $\epsilon \neq 0$ maps each solution, $u_n = c$ to $\hat{u}_n = c + \epsilon$, which can be written as $\hat{u}_n = \hat{c}$; $\hat{c} = c + \epsilon$. So Γ_ϵ is a Lie symmetry.

Note that n is a discrete variable that can't be changed by an arbitrarily small amount, so every one parameter local Lie group of symmetries must leave n unchanged. That is, $\hat{n} = n$ for all Lie symmetries of (3.1). The same argument applies to all difference equations.

Throughout the thesis, we restrict our attention to Lie symmetries for which $\hat{u}_n$ depends on n and u_n only, which are called Lie point symmetries and take the form

$$\hat{n} = n, \qquad \hat{u}_n = u_n + \epsilon Q(n, u_n) + O(\epsilon^2), \tag{3.3}$$

where $Q(n, u_n)$ is a function of n and u_n that depends on the difference equation and is called a characteristic of the local Lie group. In example (3.3), the characteristic $Q(n, u_n)$ is 1.

If we replace n by $n + k$ in equation (3.3) we get

$$\hat{u}_{n+k} = u_{n+k} + \epsilon Q(n + k, u_{n+k}) + O(\epsilon^2),$$

which is called the prolongation formula for Lie point symmetries.

In order to invest symmetries and to use them to obtain exact solutions for difference equations, we introduce the change of variable. Symmetries can also be used to simplify problems and to understand bifurcations of nonlinear difference equations.

Now consider the effect of changing variables from (n, u_n) to (n, s_n), and as (3.3) is a symmetry for each ϵ sufficiently close to *zero*, we can apply Taylor's theorem about $\epsilon = 0$, to obtain

$$\begin{aligned}
\hat{s}_n &= s(\hat{n}, \hat{u}_n) \\
&= s(n, \hat{u}_n) \\
&= s(n, u_n + \epsilon Q(n, u_n) + O(\epsilon^2)) \qquad \text{Now apply Taylor's theorem about } \epsilon = 0 \\
&= s(n, u_n + \epsilon Q(n, u_n)) \mid_{\epsilon=0} + (\epsilon - 0)\frac{ds}{d\epsilon} \mid_{\epsilon=0} + O(\epsilon^2) \\
&= s(n, u_n) + \epsilon (\frac{ds}{d\hat{u}_n})(\frac{d\hat{u}_n}{d\epsilon}) \mid_{\epsilon=0} + O(\epsilon^2) \\
&= s(n, u_n) + \epsilon s'(n, u_n) Q(n, u_n) + O(\epsilon^2).
\end{aligned}$$

If we denote the characteristic function with respect to (n, s_n) by $\hat{Q}(n, s_n)$ then we get

$$\begin{aligned}\hat{s}_n &= s_n + \epsilon\hat{Q}(n, s_n) + O(\epsilon^2) \\ &= s(n, u_n) + \epsilon s'(n, u_n)Q(n, u_n) + O(\epsilon^2).\end{aligned}$$

So we get:

$$\hat{Q}(n, s_n) = s'(n, u_n)Q(n, u_n). \tag{3.4}$$

The coordinate s_n is called the canonical coordinate.

Example 3.4. *[7] Consider changing the coordinates from (n, u_n) to (n, s_n), and symmetries for s_n,*

$$(\hat{n}, \hat{s}_n) = (n, s_n + \epsilon), \qquad \epsilon \in \mathbb{R}.$$

Then the characteristic with respect to (n, s_n) is $\hat{Q}(n, s_n) = 1$, so by (3.4),

$$s'(n, u_n)Q(n, u_n) = 1,$$

which implies that

$$s(n, u_n) = \int \frac{du_n}{Q(n, u_n)} \tag{3.5}$$

Now, as an example if $Q(n, u_n) = u_n - 1$, then the canonical coordinate according to equation (3.5) is

$$s(n, u_n) = \int \frac{du_n}{u_n - 1} = \begin{cases} \ln(u_n - 1), & |u_n| > 1 \\ \ln(1 - u_n), & |u_n| < 1 \end{cases}$$

In this example, the map from u_n to s_n isn't injective; it can't be inverted from s_n to u_n except if we specify whether $|u_n|$ is greater or less than 1.

3.2 Lie Symmetries Of A Given First Order Difference Equation

In this section, we want to solve a given first order difference equation

$$u_{n+1} = w(n, u_n), \tag{3.6}$$

by a one parameter local Lie group of symmetries.

For any transformation of a difference equation to be a symmetry, the set of solutions must be mapped to itself so the symmetry condition of equation (3.6) must be satisfied

$$\hat{u}_{n+1} = w(\hat{n}, \hat{u}_n) \quad \text{when} \quad u_{n+1} = w(n, u_n). \tag{3.7}$$

From the symmetry condition (3.7), we get

$$\begin{aligned} \hat{w}(n, u_n) &= w(\hat{n}, \hat{u}_n) \\ &= w(n, u_n + \epsilon Q(n, u_n) + O(\epsilon^2)) \\ &= w(n, u_n) + \epsilon w'(n, u_n) Q(n, u_n) + O(\epsilon^2). \end{aligned}$$

Also, we have

$$\hat{w}(n, u_n) = \hat{u}_{n+1} = u_{n+1} + \epsilon Q(n+1, u_{n+1}) + O(\epsilon^2).$$

So,

$$Q(n+1, u_{n+1}) = w'(n, u_n) Q(n, u_n). \tag{3.8}$$

This is called the linearized symmetry condition (LSC) for the given difference equation (3.6).

The linearized symmetry condition (3.8) is a linear functional equation which is difficult to solve.

Example 3.5. *[7] The linearized symmetry condition for the equation*

$$u_{n+1} - u_n = 0,$$

is

$$Q(n+1, u_{n+1}) = Q(n, u_n),$$

since $u_{n+1} = u_n$, the LSC is equivalent to

$$Q(n+1, u_{n+1}) = Q(n+1, u_n) = Q(n, u_n).$$

This condition has the general solution

$$Q(n, u_n) = f(u_n),$$

where f is an arbitrary function.

We can find the general solution of the linearized symmetry condition if we can solve the functional equation (3.8). But some functional equations can't be solved. However, there is no need to find the general solution of the linearized symmetry condition, as a single nonzero solution of this condition is sufficient to determine the general solution of the difference equation. For first order difference equations, a practical approach is to use an ansatz (trial solution) as a general solution of the linearized symmetry condition. Many physically important Lie point symmetries have characteristics of the form:

$$Q(n, u_n) = c_1(n)u_n^2 + c_2(n)u_n + c_3(n), \tag{3.9}$$

where $c_1(n)$, $c_2(n)$ and $c_3(n)$ are functions of n. By substituting (3.9) into the linearized symmetry condition (3.8) and comparing powers of u_n, we obtain the coefficients $c_1(n)$, $c_2(n)$ and $c_3(n)$.

Example 3.6. *[7] Determine the Lie point symmetries of*

$$u_{n+1} = \frac{u_n}{1 + nu_n}; \quad n \geq 1. \tag{3.10}$$

Solution. In this example, $w(n, u_n) = \frac{u_n}{1+nu_n}$. Hence,

$$w'(n, u_n) = \frac{1}{(1 + nu_n)^2},$$

so the linearized symmetry condition is

$$Q(n + 1, u_{n+1}) = \frac{1}{(1 + nu_n)^2} Q(n, u_n).$$

Since we have a first order difference equation, we can use the ansatz (3.9), to get

$$c_1(n+1)u_{n+1}^2 + c_2(n+1)u_{n+1} + c_3(n+1) = \frac{1}{(1 + nu_n)^2}(c_1(n)u_n^2 + c_2(n)u_n + c_3(n)), \tag{3.11}$$

substituting $u_{n+1} = \frac{u_n}{1+nu_n}$, we get

$$c_1(n+1)\frac{u_n^2}{(1 + nu_n)^2} + c_2(n+1)\frac{u_n}{(1 + nu_n)} + c_3(n+1) = \frac{1}{(1 + nu_n)^2}(c_1(n)u_n^2 + c_2(n)u_n + c_3(n)).$$

Multiplying the last equation by $(1 + nu_n)^2$

$$c_1(n + 1)u_n^2 + c_2(n + 1)(1 + nu_n)u_n + c_3(n + 1)(1 + nu_n)^2 = c_1(n)u_n^2 + c_2(n)u_n + c_3(n),$$

hence,

$$c_1(n+1)u_n^2 + c_2(n+1)u_n + nc_2(n+1)u_n^2 + c_3(n+1) + 2nc_3(n+1)u_n + n^2 c_3(n+1)u_n^2 = c_1(n)u_n^2 + c_2(n)u_n + c_3(n).$$

Now, comparing powers of u_n, we obtain

$$c_1(n+1) + nc_2(n+1) + n^2c_3(n+1) = c_1(n), \tag{3.12}$$

$$c_2(n+1) + 2nc_3(n+1) = c_2(n), \tag{3.13}$$

$$c_3(n+1) = c_3(n). \tag{3.14}$$

We solve this system by backward substitution, starting by equation (3.14), which is a first order linear difference equation whose solution is

$$c_3(n) = \alpha_1,$$

where α_1 is a constant. Substitute for $c_3(n)$ in equation (3.13), we obtain the first order linear difference equation

$$c_2(n+1) - c_2(n) = -2n\alpha_1,$$

which has the general solution using formula (2.7)

$$\begin{aligned} c_2(n) &= \alpha_2 - \sum_{i=0}^{n-1}(2\alpha_1 i) \\ &= \alpha_2 - 2\alpha_1\frac{n(n-1)}{2} \\ &= \alpha_2 - \alpha_1 n(n-1), \end{aligned}$$

where α_1, $\alpha_2 \in \mathbb{R}$.

Now, we substitute for $c_1(n)$ and $c_2(n)$ into equation (3.12). We get the first order difference equation

$$c_1(n+1) - c_1(n) = -\alpha_2 n + \alpha_1 n^3,$$

which has the general solution

$$\begin{aligned} c_1(n) &= \alpha_3 - \sum_{i=0}^{n-1}(\alpha_2 i) + \sum_{i=0}^{n-1}(\alpha_1 i^3) \\ &= \alpha_3 - \alpha_2\frac{n(n-1)}{2} + \alpha_1\frac{n^2(n-1)^2}{4}, \end{aligned}$$

where α_1, α_2 and $\alpha_3 \in \mathbb{R}$. So the characteristic is

$$Q(n,u_n) = \left(\alpha_3 - \alpha_2\frac{n(n-1)}{2} + \alpha_1\frac{n^2(n-1)^2}{4}\right)u_n^2 + \left(\alpha_2 - \alpha_1 n(n-1)\right)u_n + \alpha_1.$$

■

Now, we know how to find a characteristic of first order difference equations, the remaining question is how can we use a characteristic to determine the general solution of the difference equation.
Consider the canonical coordinate (3.4), and as in example (3.4) let $\hat{Q}(n, u_n) = 1$, then

$$s_n = \int \frac{du_n}{Q(n, u_n)}.$$

To use a canonical coordinate to simplify or solve a given difference equation, firstly, we write the difference equation as a difference equation for s_n, then if we can solve this equation, it remains to write the solution in terms of the original variables, and this happens only if we can invert the map from u_n to s_n . This condition is called compatibility condition, and s_n is called a compatible canonical coordinate.

Example 3.7. *[7] Find the general solution of equation (3.10) in example (3.6) using Lie symmetry*

$$u_{n+1} = \frac{u_n}{1 + nu_n}.$$

Solution. As we have found a characteristic $Q(n, u_n)$, we suppose $\alpha_1 = 0$, $\alpha_2 = 0$ and $\alpha_3 = 1$ for ease of computation.
So we obtain

$$Q(n, u_n) = u_n^2.$$

The canonical coordinate

$$s_n = \int \frac{du_n}{u_n^2} = \frac{-1}{u_n},$$

which is compatible since we can write u_n in terms of s_n. Now we consider the difference equation

$$s_{n+1} - s_n = \frac{-1}{u_{n+1}} - \frac{-1}{u_n},$$

if we substitute $u_{n+1} = \frac{u_n}{1+nu_n}$, we get:

$$s_{n+1} - s_n = -n,$$

which has the general solution:

$$s_n = c - \frac{n(n-1)}{2}; \; c \in \mathbb{R},$$

but $s_n = \frac{-1}{u_n}$, so the general solution of the original difference equation is

$$u_n = \frac{2}{-2c + n(n-1)}; c \in \mathbb{R}.$$

■

3.3 Symmetries And Second Order Difference Equations

The linearized symmetry condition (LSC) for second order difference equations is given by the same way as that for the first order difference equations.
Now, consider the difference equation

$$u_{n+2} = w(n, u_n, u_{n+1}); \quad n \in \mathbb{Z}, \tag{3.15}$$

we assume that $\frac{\partial w}{\partial u_{n+1}} \neq 0$, (this condition ensures that the equation is truely second order), the symmetry condition is

$$\hat{u}_{n+2} = w(\hat{n}, \hat{u}_n, \hat{u}_{n+1}), \quad \text{when (3.15) holds.} \tag{3.16}$$

As before, we restrict our attention to Lie symmetries of the form

$$\hat{n} = n, \quad \hat{u}_{n+p} = u_{n+p} + \epsilon Q(n+p, u_{n+p}) + O(\epsilon^2),$$

substitute into (3.16) to get

$$w(\hat{n}, \hat{u}_n, \hat{u}_{n+1}) = w(n, u_n + \epsilon Q(n, u_n), u_{n+1} + \epsilon Q(n+1, u_{n+1})). \tag{3.17}$$

Find Taylor series of the right hand side about $\epsilon = 0$, we get

$$\begin{aligned} w(\hat{n}, \hat{u}_n, \hat{u}_{n+1}) &= w(n, u_n, u_{n+1}) + \epsilon \left(\frac{\partial w}{\partial \hat{u}_{n+1}} \frac{\partial \hat{u}_{n+1}}{\partial \epsilon} |_{\epsilon=0} + \frac{\partial w}{\partial \hat{u}_n} \frac{\partial \hat{u}_n}{\partial \epsilon} |_{\epsilon=0} \right) + O(\epsilon^2) \\ &= w(n, u_n, u_{n+1}) + \epsilon \left(\frac{\partial w}{\partial u_{n+1}} Q(n+1, u_{n+1}) + \frac{\partial w}{\partial u_n} Q(n, u_n) \right) + O(\epsilon^2), \end{aligned} \tag{3.18}$$

also we have

$$w(\hat{n}, \hat{u}_n, \hat{u}_{n+1}) = \hat{u}_{n+2} = w(n, u_n, u_{n+1}) + \epsilon Q(n+2, u_{n+2}) + O(\epsilon^2). \tag{3.19}$$

From equation (3.18) and (3.19), we get the linearized symmetry condition (LSC) for second order difference equations

$$Q(n+2, u_{n+2}) = \frac{\partial w}{\partial u_{n+1}} Q(n+1, u_{n+1}) + \frac{\partial w}{\partial u_n} Q(n, u_n)$$

To simplify this formula, we introduce the definition of the infinitesimal generator.

Definition 3.3.1. *[8] The infinitesimal generator X is*

$$X = \sum_{k=0}^{p-1} (S^k Q(n, u_n)) \frac{\partial}{\partial u_{n+k}},$$

where S^k is the forward shift operator such that $S^k u_n = u_{n+k}$ and p is the order of the difference equation.

So the Linearized symmetry condition for second order difference equations can be written as

$$S^2 Q - Xw = 0, \tag{3.20}$$

which is a linear functional equation for the characteristics $Q(n, u_n)$. However, functional equation are generally hard to solve. Lie symmetries are diffeomorphisms, that is, $Q(n, u_n)$ is a smooth function, so the linearized symmetry condition can be solved by the method of differential elimination.

To explain the steps that transform equation (3.20) from a functional equation to a differential equation, we consider the difference equations that satisfy the conditions $\frac{\partial w}{\partial u_{n+1}} \neq 0$ and $\frac{\partial w}{\partial u_n} \neq 0$. We follow the steps

Firstly, by eliminating $Q(n+2, w)$ and $Q(n+1, u_{n+1})$, we can form an ordinary differential equation for $Q(n, u_n)$. To achieve this objective we differentiate the linearized symmetry condition with respect to u_n keeping w fixed and we consider u_{n+1} to be a function of n, u_n and w. Therefore, we apply the differential operator (L):

$$L = \frac{\partial}{\partial u_n} + \frac{\partial u_{n+1}}{\partial u_n} \frac{\partial}{\partial u_{n+1}},$$

but

$$\frac{\partial u_{n+1}}{\partial u_n} = -\frac{\partial w / \partial u_n}{\partial w / \partial u_{n+1}}.$$

The first term of the functional equation (3.20) is eliminated by this differential operator, since we differentiate with respect to u_n keeping w fixed, so we obtain

$$\begin{aligned}
\frac{\partial}{\partial u_n}\left(Q(n+2, w)\right) &= 0, \\
\frac{\partial}{\partial u_n}\left(\frac{\partial w}{\partial u_n} Q(n, u_n)\right) &= \frac{\partial w}{\partial u_n} Q'(n, u_n) + \frac{\partial^2 w}{\partial u_n^2} Q(n, u_n), \\
\frac{\partial}{\partial u_n}\left(\frac{\partial w}{\partial u_{n+1}} Q(n+1, u_{n+1})\right) &= \frac{\partial^2 w}{\partial u_n \partial u_{n+1}} Q(n+1, u_{n+1}),
\end{aligned}$$

and

$$\frac{\partial}{\partial u_{n+1}}\Bigg(Q(n+2,w)\Bigg)=0,$$
$$\frac{\partial}{\partial u_{n+1}}\Bigg(\frac{\partial w}{\partial u_n}Q(n,u_n)\Bigg)=\frac{\partial^2 w}{\partial u_{n+1}\partial u_n}Q(n,u_n),$$
$$\frac{\partial}{\partial u_{n+1}}\Bigg(\frac{\partial w}{\partial u_{n+1}}Q(n+1,u_{n+1})\Bigg)=\frac{\partial w}{\partial u_{n+1}}Q'(n+1,u_{n+1})+\frac{\partial^2 w}{\partial u_{n+1}^2}Q(n+1,u_{n+1}).$$

This implies that

$$\Bigg(-\frac{\partial w}{\partial u_n}Q'(n,u_n)-\frac{\partial^2 w}{\partial u_n^2}Q(n,u_n)-\frac{\partial^2 w}{\partial u_n\partial u_{n+1}}Q(n+1,u_{n+1})\Bigg)+$$
$$\Bigg(\frac{\partial u_{n+1}}{\partial u_n}\Bigg)\Bigg(-\frac{\partial^2 w}{\partial u_{n+1}\partial u_n}Q(n,u_n)-\frac{\partial w}{\partial u_{n+1}}Q'(n+1,u_{n+1})-\frac{\partial^2 w}{\partial u_{n+1}^2}Q(n+1,u_{n+1})\Bigg)=0.$$

Secondly, we can eliminate $Q'(n+1,u_{n+1})$ by differentiating the equation obtained in the previous step with respect to u_n keeping u_{n+1} fixed. We may have to differentiate once more with respect to u_n keeping u_{n+1} fixed. After that, we obtain an ordinary differential equation, which can be split by gathering together all terms with the same dependence upon u_{n+1} and we solve it if possible, and obtain $Q(n,u_n)$. To find the coefficients of the terms of $Q(n,u_n)$, we plug it in the equations that we obtained in previous steps which can be split into a system of linear difference equations by collecting all terms with the same dependence u_n and u_{n+1}. Example (3.8) illustrates this method.

Example 3.8. *Find the characteristics of the equation:*

$$u_{n+2}=\frac{au_nu_{n+1}}{u_n+u_{n+1}};\quad a\in\mathbb{R}-\{0\}.$$

Solution. The *LSC* is

$$Q(n+2,u_{n+2})-\frac{\partial w}{\partial u_n}Q(n,u_n)-\frac{\partial w}{\partial u_{n+1}}Q(n+1,u_{n+1})=0,$$

but

$$\frac{\partial w}{\partial u_n}=\frac{au_{n+1}^2}{(u_n+u_{n+1})^2}=\frac{w^2}{au_n^2},$$

and

$$\frac{\partial w}{\partial u_{n+1}}=\frac{au_n^2}{(u_n+u_{n+1})^2}=\frac{w^2}{au_{n+1}^2},$$

so

$$\frac{\partial u_{n+1}}{\partial u_n}=-\frac{u_{n+1}^2}{u_n^2},$$

so the LSC is

$$Q(n+2,u_{n+2}) - \frac{w^2}{au_n^2}Q(n,u_n) - \frac{w^2}{au_{n+1}^2}Q(n+1,u_{n+1}) = 0. \qquad (3.21)$$

To transform this functional equation to differential equation, we apply the differential operator (L)

$$\begin{aligned} L &= \frac{\partial}{\partial u_n} + \frac{\partial u_{n+1}}{\partial u_n}\frac{\partial}{\partial u_{n+1}} \\ &= \frac{\partial}{\partial u_n} - \frac{u_{n+1}^2}{u_n^2}\frac{\partial}{\partial u_{n+1}}, \end{aligned}$$

so we get

$$(\frac{\partial}{\partial u_n} - \frac{u_{n+1}^2}{u_n^2}\frac{\partial}{\partial u_{n+1}})(Q(n+2,u_{n+2}) - \frac{w^2}{au_n^2}Q(n,u_n) - \frac{w^2}{au_{n+1}^2}Q(n+1,u_{n+1})) = 0,$$

but

$$\begin{aligned} &\frac{\partial}{\partial u_n}(Q(n+2,u_{n+2})) = 0, \\ &\frac{\partial}{\partial u_n}(\frac{w^2}{au_n^2}Q(n,u_n)) = \frac{w^2}{au_n^2}Q'(n,u_n) + \frac{-2w^2}{au_n^3}Q(n,u_n), \\ &\frac{\partial}{\partial u_n}(\frac{w^2}{au_{n+1}^2}Q(n+1,u_{n+1})) = 0, \end{aligned}$$

and

$$\begin{aligned} &\frac{\partial}{\partial u_{n+1}}(Q(n+2,u_{n+2})) = 0, \\ &\frac{\partial}{\partial u_{n+1}}(\frac{w^2}{au_n^2}Q(n,u_n)) = 0, \\ &\frac{\partial}{\partial u_{n+1}}(\frac{w^2}{au_{n+1}^2}Q(n+1,u_{n+1})) = \frac{w^2}{au_{n+1}^2}Q'(n+1,u_{n+1}) + \frac{-2w^2}{au_{n+1}^3}Q(n+1,u_{n+1}), \end{aligned}$$

this implies:

$$\frac{-w^2}{au_n^2}Q'(n,u_n) + \frac{2w^2}{au_n^3}Q(n,u_n) - \frac{u_{n+1}^2}{u_n^2}(\frac{-w^2}{au_{n+1}^2}Q'(n+1,u_{n+1}) + \frac{2w^2}{au_{n+1}^3}Q(n+1,u_{n+1})) = 0,$$

multiplying the last equation by $\frac{-au_n^2}{w^2}$, we get

$$Q'(n,u_n) - \frac{2}{u_n}Q(n,u_n) - Q'(n+1,u_{n+1}) + \frac{2}{u_{n+1}}Q(n+1,u_{n+1}) = 0, \qquad (3.22)$$

now, we differentiate (3.22) with respect to u_n keeping u_{n+1} fixed, we obtain

$$\frac{\partial}{\partial u_n}(Q'(n,u_n) - \frac{2}{u_n}Q(n,u_n) - Q'(n+1,u_{n+1}) + \frac{2}{u_{n+1}}Q(n+1,u_{n+1})) = 0,$$

but

$$\begin{aligned}
&\frac{\partial}{\partial u_n}(Q'(n,u_n)) = Q''(n,u_n),\\
&\frac{\partial}{\partial u_n}(\frac{2}{u_n}Q(n,u_n)) = \frac{2}{u_n}Q'(n,u_n) + \frac{-2}{u_n^2}Q(n,u_n),\\
&\frac{\partial}{\partial u_n}(Q'(n+1,u_{n+1})) = 0,\\
&\frac{\partial}{\partial u_n}(\frac{2}{u_{n+1}}Q(n+1,u_{n+1})) = 0,
\end{aligned}$$

so

$$Q''(n,u_n) - \frac{2}{u_n}Q'(n,u_n) + \frac{2}{u_n^2}Q(n,u_n) = 0,$$

if we multiply this equation by u_n^2, we get

$$u_n^2 Q''(n,u_n) - 2u_n Q'(n,u_n) + 2Q(n,u_n) = 0,$$

which is an Euler differential equation whose solution is given by

$$Q(n,u_n) = \alpha(n)u_n^2 + \beta(n)u_n,$$

for some functions α and β of n. Substituting $Q(n,u_n)$ into (3.22) gives

$$\begin{aligned}
2\alpha(n)u_n + \beta(n) - \frac{2}{u_n}(\alpha(n)u_n^2 + \beta(n)u_n) - 2\alpha(n+1)u_{n+1} - \beta(n+1) + \\
\frac{2}{u_{n+1}}(\alpha(n+1)u_{n+1}^2 + \beta(n+1)u_{n+1}) = 0,
\end{aligned}$$

simplifying, we get

$$-\beta(n) + \beta(n+1) = 0,$$

which is a first order linear difference equation whose solution is

$$\beta(n) = c,$$

where $c \in \mathbb{R}$.

Now, we substitute $Q(n,u_n) = \alpha(n)u_n^2 + cu_n$, in (3.21) to obtain

$$\alpha(n+2)u_{n+2}^2 + cu_{n+2} - \frac{w^2}{au_n^2}(\alpha(n)u_n^2 + cu_n) - \frac{w^2}{au_{n+1}^2}(\alpha(n+1)u_{n+1}^2 + cu_{n+1}) = 0, \quad (3.23)$$

but

$$
\begin{aligned}
cu_{n+2} - c\frac{w^2}{au_n} - c\frac{w^2}{au_{n+1}} &= cw - cw\frac{u_{n+1}}{u_n + u_{n+1}} - cw\frac{u_n}{u_n + u_{n+1}} \\
&= cw(1 - \frac{u_{n+1}}{u_n + u_{n+1}} - \frac{u_n}{u_n + u_{n+1}}) \\
&= 0,
\end{aligned}
$$

so equation (3.23) simplifies to

$$\alpha(n+2)w^2 - \frac{1}{a}\alpha(n+1)w^2 - \frac{1}{a}\alpha(n)w^2 = 0,$$

this implies

$$\alpha(n+2) - \frac{1}{a}\alpha(n+1) - \frac{1}{a}\alpha(n) = 0,$$

which is a second order linear difference equation and has the characteristic equation

$$r^{n+2} - \frac{1}{a}r^{n+1} - \frac{1}{a}r^n = 0,$$

which implies that

$$r^2 - \frac{1}{a}r - \frac{1}{a} = 0,$$

so the characteristic roots are

$$r = \frac{1}{2a} + \frac{1}{2|a|}\sqrt{1+4a} \quad \text{and} \quad r = \frac{1}{2a} - \frac{1}{2|a|}\sqrt{1+4a}.$$

Hence, we have the following cases:

1. if $a = \frac{-1}{4}$, then

   $$\alpha(n) = c_1(-2)^n + c_2 n(-2)^n,$$

   where c_1 and $c_2 \in \mathbb{R}$.
   So the characteristic is

   $$Q(n, u_n) = (c_1(-2)^n + c_2 n(-2)^n)u_n^2 + cu_n,$$

   where c, c_1 and $c_2 \in \mathbb{R}$.

2. $if a \neq \frac{-1}{4}$, then

   $$\alpha(n) = c_1(\frac{1}{2a} + \frac{1}{2|a|}\sqrt{1+4a})^n + c_2(\frac{1}{2a} - \frac{1}{2|a|}\sqrt{1+4a})^n,$$

where c_1 and $c_2 \in \mathbb{R}$.
So the characteristic is

$$Q(n, u_n) = (c_1(\frac{1}{2a} + \frac{1}{2|a|}\sqrt{1+4a})^n + c_2(\frac{1}{2a} - \frac{1}{2|a|}\sqrt{1+4a})^n)u_n^2 + cu_n,$$

where c, c_1 and $c_2 \in \mathbb{R}$.

■

Now, to invest symmetries in reducing the order of difference equations, we find a compatible canonical coordinate, which reduces the order by one. If the reduced equation can be solved, then the original equation can be solved by one more integration or summation.

Definition 3.3.2. *[4] A function v_n is invariant under the Lie group of transformations Γ_a if $Xv_n = 0$, where $X = \sum_{k=0}^{p-1} S^k Q(n, u_n) \frac{\partial}{\partial u_{n+k}}$.*

Suppose that the characteristic $Q(n, u_n)$ for the second order difference equation

$$u_{n+2} = w(n, u_n, u_{n+1}),$$

is known, then the invariant v_n can be found by solving the partial differential equation

$$Xv_n = Q(n, u_n)\frac{\partial v_n}{\partial u_n} + Q(n+1, u_{n+1})\frac{\partial v_n}{\partial u_{n+1}} = 0,$$

which is a quasi linear partial differential equation that can be solved using the method of characteristics, set

$$\frac{du_n}{Q(n, u_n)} = \frac{du_{n+1}}{SQ(n, u_n)} = \frac{dv_n}{0}. \tag{3.24}$$

If the invariant function $v_{n+1}(n, u_n, u_{n+1})$ can be written as a function of n and v_n only, then v_n can reduce the order of the difference equation by *one* to obtain

$$u_{n+1} = f(n, u_n, v_n),$$

for some function f. This equation is a first order difference equation.
Finally, as we mentioned in the previous section, to solve the first order equation, we need to obtain a canonical coordinate s_n.

Example 3.9. *[7] Consider the equation in Example (3.8), with $a = 2$,*

$$u_{n+2} = \frac{2u_n u_{n+1}}{u_n + u_{n+1}}.$$

Solution. We have seen for $a = 2$, the characteristic is

$$Q(n, u_n) = (c_1 + c_2(\frac{-1}{2})^n)u_n^2 + cu_n.$$

To simplify calculations, take $c_1 = 1$, $c_2 = 0$ and $c_3 = 0$, we obtain

$$Q(n, u_n) = u_n^2.$$

So the canonical coordinate is

$$s_n = \int \frac{du_n}{u_n^2} = \frac{-1}{u_n}.$$

By equation (3.24), the invariant v_n is given by

$$\frac{du_n}{u_n^2} = \frac{du_{n+1}}{u_{n+1}^2} = \frac{dv_n}{0}.$$

Taking the first $\left(\frac{du_n}{u_n^2}\right)$ and second $\left(\frac{du_{n+1}}{u_{n+1}^2}\right)$ invariants, we get

$$\frac{-1}{u_{n+1}} = \frac{-1}{u_n} + c_1, \quad \text{which implies} \quad c_1 = \frac{-1}{u_{n+1}} - \frac{-1}{u_n},$$

where $c_1 \in \mathbb{R}$. Also, we have

$$\frac{du_n}{u_n^2} = \frac{dv_n}{0},$$

which implies that

$$v_n = c_2, \quad \text{such that} \quad c_2 = f(c_1),$$

where c_1 and c_2 are constants, and f is an arbitrary function which we take to be the identity function,so

$$f(c_1) = c_1 \Rightarrow c_2 = c_1$$

therefore

$$v_n = c_2 = \frac{1}{u_n} - \frac{1}{u_{n+1}}. \tag{3.25}$$

Applying the shift operator to v_n, we get

$$\begin{aligned} v_{n+1} &= \frac{1}{u_{n+1}} - \frac{1}{u_{n+2}} \\ &= \frac{1}{u_{n+1}} - \frac{u_{n+1} + u_n}{2u_n u_{n+1}} \\ &= \frac{1}{2u_{n+1}} - \frac{1}{2u_n} \\ &= -\frac{v_n}{2}. \end{aligned}$$

So, we have the equation

$$v_{n+1}+\frac{v_n}{2}=0,$$

which is a first order linear difference equation whose solution is given by

$$v_n=\alpha\left(\frac{-1}{2}\right)^n,$$

where $\alpha \in \mathbb{R}$. It follows that

$$s_{n+1}-s_n=\frac{-1}{u_{n+1}}-\frac{-1}{u_n}=v_n=\alpha\left(\frac{-1}{2}\right)^n,$$

this equation is a first order linear difference equation whose solution is given by

$$\begin{aligned} s_n=s_0+\sum_{k=0}^{n-1}\alpha\left(\frac{-1}{2}\right)^k &= s_0+\alpha\frac{(1-(\frac{-1}{2})^n)}{1-\frac{-1}{2}} \\ &= s_0+\alpha\frac{2(1-(\frac{-1}{2})^n)}{3}, \end{aligned} \tag{3.26}$$

but $s_n=\frac{-1}{u_n}$, so

$$\begin{aligned} u_n &= \frac{-1}{s_0+\alpha\frac{2(1-(\frac{-1}{2})^n)}{3}} \\ &= \frac{-1}{\frac{-1}{u_0}+\alpha\frac{2(1-(\frac{-1}{2})^n)}{3}} \\ &= \frac{1}{\frac{1}{u_0}-\frac{2}{3}\alpha(1-(\frac{-1}{2})^n)} \\ &= \frac{1}{(\frac{1}{u_0}-\frac{2\alpha}{3})+\frac{2\alpha}{3}(-2)^{-n}} \\ &= \frac{1}{\hat{c}_1+\hat{c}_2(-2)^{-n}}, \end{aligned}$$

where $\hat{c}_1$ and $\hat{c}_2 \in \mathbb{R}$, and they are not both *zero*. ■

Example 3.10. *[5] Find the exact solution of the difference equation*

$$u_{n+2}=\frac{au_n}{1+bu_nu_{n+1}}. \tag{3.27}$$

Solution. The linearized symmetry condition *LSC* to equation (3.27) is

$$Q(n+2,w)-\frac{\partial w}{\partial u_n}Q(n,u_n)-\frac{\partial w}{\partial u_{n+1}}Q(n+1,u_{n+1})=0,$$

but,

$$\frac{\partial w}{\partial u_n} = \frac{a}{(1+bu_nu_{n+1})^2} = \frac{w^2}{au_n^2},$$

and

$$\frac{\partial w}{\partial u_{n+1}} = \frac{-abu_n^2}{(1+bu_nu_{n+1})^2} = \frac{-bw^2}{a},$$

so the LSC is given by

$$Q(n+2,w) - \frac{w^2}{au_n^2}Q(n,u_n) + \frac{bw^2}{a}Q(n+1,u_{n+1}) = 0. \tag{3.28}$$

Firstly, we apply the differential operator L, given by

$$L = \frac{\partial}{\partial u_n} + \frac{1}{bu_n^2}\frac{\partial}{\partial u_{n+1}},$$

to equation (3.28) to get

$$\frac{\partial}{\partial u_n}\left(Q(n+2,w) - \frac{w^2}{au_n^2}Q(n,u_n) + \frac{bw^2}{a}Q(n+1,u_{n+1})\right) + \left(\frac{1}{bu_n^2}\frac{\partial}{\partial u_{n+1}}\right)\left(Q(n+2,w) - \frac{w^2}{au_n^2}Q(n,u_n) + \frac{bw^2}{a}Q(n+1,u_{n+1})\right) = 0,$$

but

$$\begin{aligned}
\frac{\partial}{\partial u_n}\left(Q(n+2,w)\right) &= 0,\\
\frac{\partial}{\partial u_n}\left(\frac{w^2}{au_n^2}Q(n,u_n)\right) &= \frac{1}{au_n^2}w^2Q'(n,u_n) + \frac{-2}{au_n^3}w^2Q(n,u_n),\\
\frac{\partial}{\partial u_n}\left(\frac{b}{a}w^2Q(n+1,u_{n+1})\right) &= 0,\\
\frac{\partial}{\partial u_{n+1}}\left(Q(n+2,w)\right) &= 0,\\
\frac{\partial}{\partial u_{n+1}}\left(\frac{w^2}{au_n^2}Q(n,u_n)\right) &= 0,\\
\frac{\partial}{\partial u_{n+1}}\left(\frac{b}{a}w^2Q(n+1,u_{n+1})\right) &= \frac{b}{a}w^2Q'(n+1,u_{n+1}),
\end{aligned}$$

this leads to

$$\frac{-1}{au_n^2}w^2Q'(n,u_n) + \frac{2}{au_n^3}w^2Q(n,u_n) + \frac{1}{au_n^2}Q'(n+1,u_{n+1}) = 0,$$

multiplying this equation by $\frac{-au_n^2}{w^2}$, we get

$$Q'(n,u_n) - \frac{2}{u_n}Q(n,u_n) - Q'(n+1,u_{n+1}) = 0, \tag{3.29}$$

now, we differentiate equation (3.29) with respect to u_n keeping u_{n+1} fixed. As a result we obtain the ODE

$$Q''(n,u_n) - \frac{2}{u_n}Q'(n,u_n) + \frac{2}{u_n^2}Q(n,u_n) = 0,$$

which is a Cauchy differential equation, whose solution is given by

$$Q(n,u_n) = \alpha(n)u_n^2 + \beta(n)u_n. \tag{3.30}$$

Next we substitute (3.30) into (3.29), we get

$$2\alpha(n)u_n + \beta(n) - 2\alpha(n+1)u_{n+1} - \beta(n+1) - 2\alpha(n)u_n - 2\beta(n) = 0,$$

the equation can be split by gathering together all terms with the same dependence upon u_{n+1}

$$-2\alpha(n+1)u_{n+1} - (\beta(n+1) + \beta(n)) = 0.$$

Now, we compare the two sides of the last equation, to obtain

$$\beta(n+1) + \beta(n) = 0,$$

which is a first order linear difference equation whose general solution is

$$\beta(n) = c(-1)^n,$$

where c is a constant. and

$$\alpha(n+1) = 0 \text{ which implies } \alpha(n) = 0.$$

So

$$Q(n,u_n) = (-1)^n u_n.$$

We want to find the invariant using equation (3.24),

$$\frac{du_n}{(-1)^n u_n} = \frac{du_{n+1}}{(-1)^{n+1}u_{n+1}} = \frac{dv_n}{0},$$

Taking the first $\left(\frac{du_n}{(-1)^n u_n}\right)$ and second $\left(\frac{du_{n+1}}{(-1)^{n+1} u_{n+1}}\right)$ invariants, we get

$$\ln|u_n| + c^* = -\ln|u_{n+1}| \quad \text{which implies} \quad -c^* = \ln|u_{n+1}u_n|,$$

where $c^* \in \mathbb{R}$, so

$$k_1 = u_n u_{n+1} \quad \text{where} \quad k_1 = e^{-c^*},$$

also, we have

$$\frac{du_n}{u_n} = \frac{dv_n}{0},$$

which implies that

$$v_n = k, \quad \text{such that} \quad k = f(k_1),$$

where k_1 and k are constants.
We choose $f(k_1) = k_1$, therefore

$$v_n = u_n u_{n+1}. \tag{3.31}$$

Applying the shift operator to v_n yields

$$\begin{aligned} Sv_n = v_{n+1} &= u_{n+1}u_{n+2} \\ &= u_{n+1}\left(\frac{au_n}{1 + bu_n u_{n+1}}\right) \\ &= \frac{av_n}{1 + bv_n}, \end{aligned} \tag{3.32}$$

So we have the equation

$$v_{n+1} = \frac{av_n}{1 + bv_n},$$

which is a Riccati difference equation of type one, where $g(n) = 0$ so to solve it we let

$$z_n = \frac{1}{v_n},$$

we get

$$z_{n+1} - \frac{1}{a}z_n - \frac{b}{a} = 0,$$

which is a linear difference equation, whose solution is given by

$$z_n = \begin{cases} z_0 + nb; & a = 1 \\ \frac{z_0}{a^n} + \frac{b}{a^n(1-a)} + \frac{b}{a-1}; & a \neq 1 \end{cases}$$

and this implies

$$v_n = \begin{cases} \frac{1}{z_0 + nb}; & a = 1 \\ \frac{a^n(1-a)}{z_0(1-a) + b(1-a^n)}; & a \neq 1 \end{cases}$$

but $z_0 = \frac{1}{v_0}$, so

$$v_n = \begin{cases} \frac{v_0}{1+nbv_0}; & a = 1 \\ \frac{a^n(a-1)v_0}{(a-1)+bv_0(a^n-1)}; & a \neq 1 \end{cases}$$

Now, we want to consider the two cases.

If $a = 1$, we have

$$v_n = \frac{v_0}{1 + nbv_0} \tag{3.33}$$

Then by equations (3.31) and (3.33) we have

$$v_n = u_n u_{n+1} = \frac{v_0}{1 + nbv_0},$$

where $v_0 = u_0 u_1$. Solving the last equation for u_{n+1} we obtain

$$u_{n+1} = \frac{v_0}{(1 + nbv_0)u_n}. \tag{3.34}$$

The order of Equation (3.27) has been reduced by one.

To solve equation (3.34) we need to obtain a canonical coordinate,

$$\begin{aligned} s_n &= \int \frac{du_n}{(-1)^n u_n} \\ &= (-1)^n \ln|u_n|. \end{aligned}$$

So $s_{n+1} - s_n$ is an invariant. Consequently,

$$\begin{aligned} s_{n+1} - s_n &= (-1)^{n+1} \ln|u_{n+1}| - (-1)^n \ln|u_n| \\ &= (-1)^{n+1} \ln|u_n u_{n+1}| \\ &= (-1)^{n+1} \ln|v_n| \\ &= (-1)^{n+1} \ln\left|\frac{v_0}{1 + nbv_0}\right|, \end{aligned} \tag{3.35}$$

The general solution of (3.35) is

$$\begin{aligned} s_n &= s_0 + \sum_{m=0}^{n-1} (-1)^{m+1} \ln|v_m| \\ &= \ln|u_0| + \sum_{m=0}^{n-1} (-1)^{m+1} \ln\left|\frac{u_0 u_1}{1 + bu_0 u_1 m}\right|, \end{aligned}$$

but $s_n = (-1)^n \ln|u_n|$, so the general solution of (3.27) if $a = 1$ is

$$\begin{aligned} u_n &= \exp\left((-1)^n \ln|u_0| + \sum_{m=0}^{n-1}(-1)^{m+n+1}\ln\left|\frac{u_0u_1}{1+bu_0u_1m}\right|\right) \\ &= \exp\left((-1)^n \ln|u_0| + (-1)^{n+1}\ln|u_0u_1| + \sum_{m=1}^{n-1}(-1)^{m+n+1}\ln\left|\frac{u_0u_1}{1+bu_0u_1m}\right|\right) \\ &= (u_1)^{(-1)^{n+1}}\prod_{m=1}^{n-1}\left(\frac{u_0u_1}{1+bu_0u_1m}\right)^{(-1)^{m+n+1}}, \end{aligned}$$

with $1 + bu_0u_1m \neq 0$ for all $m = \{1, 2, \cdots, n-1\}$ that is
$m \neq \frac{-1}{bu_0u_1}$, for all $m = \{1, 2, \cdots, n-1\}$ which implies $\frac{-1}{bu_0u_1} \notin \{1, 2, \cdots, n-1\}$.
Now, if $a \neq 1$, we have

$$v_n = \frac{a^n(a-1)v_0}{(a-1)+bv_0(a^n-1)}. \tag{3.36}$$

The canonical coordinate is

$$\begin{aligned} s_n &= s_0 + \sum_{m=0}^{n-1}(-1)^{m+1}\ln|v_m| \\ &= \ln|u_0| + \sum_{m=0}^{n-1}(-1)^{m+1}\ln\left|\frac{a^m(a-1)v_0}{(a-1)+bv_0(a^m-1)}\right|, \end{aligned}$$

but $s_n = (-1)^n \ln|u_n|$, so the general solution of (3.27) if $a \neq 1$ is

$$\begin{aligned} u_n &= \exp\left((-1)^n \ln|u_0| + \sum_{m=0}^{n-1}(-1)^{m+n+1}\ln\left|\frac{a^m(a-1)v_0}{(a-1)+bv_0(a^m-1)}\right|\right) \\ &= (u_0)^{(-1)^n}\prod_{m=0}^{n-1}\left(\frac{a^m(a-1)u_0u_1}{(a-1)+bu_0u_1(a^m-1)}\right)^{(-1)^{m+n+1}}. \end{aligned}$$

■

3.4 Symmetries And Higher Order Difference Equations

In this section, we want to describe the method for finding Lie symmetries of a general ordinary difference equation. Consider the ordinary difference equation of order $p \geq 3$ of the form

$$u_{n+p} = w(n, u_n, u_{n+1}, ..., u_{n+p-1}); \quad \frac{\partial w}{\partial u_n} \neq 0. \tag{3.37}$$

The linearized symmetry condition for equation (3.37) is obtained by the same way as that for second order difference equations. Moreover, the same approach can be applied to find all characteristics $Q(n, u_n)$, but the calculations are more complicated, so it is

necessary to use computer algebra.
The general technique for obtaining Lie point symmetry for any difference equation of order $p \geq 2$:

1. Write down the LSC for the ordinary difference equation,

$$S^{(p)}Q(n, u_n) - Xw = 0. \tag{3.38}$$

2. Apply appropriate differential operators to reduce the number of unknown functions, then differentiate the LSC withe respect to suitable independent variable. Continue doing this until an ODE is obtained.

3. Simplify the ODE, if possible, then solve it.

4. Substitute the results into the equations obtained in step (2).

5. Solve the resulting linear difference equations.

6. Finally, substitute $Q(n, u_n)$ into the LSC, simplify any remaining difference equations, and solve it.

After finding the characteristics $Q(n, u_n)$, we want to invest symmetries to reduce the order of difference equations, as in the second order case. We find a compatible canonical coordinate s_n, which reduces the order by one. Moreover, we want to find the invariant v_n and follow a similar way to solve a higher order difference equation. For equation (3.37), we suppose the characteristic $Q(n, u_n)$ is known, then the invariant v_n can be found by solving the partial differential equation

$$Xv_n = Q(n, u_n)\frac{\partial v_n}{\partial u_n} + SQ(n, u_n)\frac{\partial v_n}{\partial u_{n+1}} + \cdots + S^{p-1}Q(n, u_n)\frac{\partial v_n}{\partial u_{n+p-1}} = 0,$$

and the general technique to solve the partial differential equations of this form is known as the method of characteristics and is useful for finding analytic solutions.
To solve these equations, we set

$$\frac{du_n}{Q(n, u_n)} = \frac{du_{n+1}}{SQ(n, u_n)} = \cdots = \frac{du_{n+p-1}}{S^{p-1}Q(n, u_n)} = \frac{dv_n}{0}. \tag{3.39}$$

Chapter 4

Applications Of Symmetry Method To Some Difference Equations

4.1 Symmetry Analysis And Exact Solution Of The Difference Equation $u_{n+2} = (n + u_n u_{n+1})/(u_{n+1})$

Consider the second order nonlinear difference equation

$$w(n, u_n, u_{n+1}) = u_{n+2} = \frac{n + u_n u_{n+1}}{u_{n+1}}. \tag{4.1}$$

We investigate the exact solution of the second order difference equation using Lie symmetries. As we mentioned earlier, we shall assume that the characteristic $Q(n, u_n)$ depends on n and u_n only and we use it to find the exact solutions.
The linearized symmetry condition LSC to equation (4.1) is

$$Q(n+2, w) - \frac{\partial w}{\partial u_n} Q(n, u_n) - \frac{\partial w}{\partial u_{n+1}} Q(n+1, u_{n+1}) = 0,$$

but

$$\frac{\partial w}{\partial u_n} = 1,$$

and

$$\frac{\partial w}{\partial u_{n+1}} = \frac{-n}{u_{n+1}^2},$$

so the LSC is

$$Q(n+2,w) - Q(n,u_n) + \frac{n}{u_{n+1}^2} Q(n+1,u_{n+1}) = 0. \tag{4.2}$$

Now, we apply the differential operator L given by

$$L = \frac{\partial}{\partial u_n} + \frac{u_{n+1}^2}{n} \frac{\partial}{\partial u_{n+1}},$$

to equation (4.2) to get

$$\frac{\partial}{\partial u_n}\left(Q(n+2,w) - Q(n,u_n) + \frac{n}{u_{n+1}^2} Q(n+1,u_{n+1}) \right) + \left(\frac{u_{n+1}^2}{n} \frac{\partial}{\partial u_{n+1}} \right)\left(Q(n+2,w) - Q(n,u_n) + \frac{n}{u_{n+1}^2} Q(n+1,u_{n+1}) \right) = 0,$$

but

$$\begin{aligned}
&\frac{\partial}{\partial u_n}\left(Q(n+2,w) \right) = 0, \\
&\frac{\partial}{\partial u_n}\left(Q(n,u_n) \right) = Q'(n,u_n), \\
&\frac{\partial}{\partial u_n}\left(\frac{n}{u_{n+1}^2} Q(n+1,u_{n+1}) \right) = 0, \\
&\frac{\partial}{\partial u_{n+1}}\left(Q(n+2,w) \right) = 0, \\
&\frac{\partial}{\partial u_{n+1}}\left(Q(n,u_n) \right) = 0, \\
&\frac{\partial}{\partial u_{n+1}}\left(\frac{n}{u_{n+1}^2} Q(n+1,u_{n+1}) \right) = \frac{n}{u_{n+1}^2} Q'(n+1,u_{n+1}) + \frac{-2n}{u_{n+1}^3} Q(n+1,u_{n+1}),
\end{aligned}$$

this leads to

$$Q'(n+1,u_{n+1}) - \frac{2}{u_{n+1}} Q(n+1,u_{n+1}) - Q'(n,u_n) = 0, \tag{4.3}$$

now, we differentiate this equation with respect to u_n keeping u_{n+1} fixed. As a result we obtain the ODE

$$-Q''(n,u_n) = 0,$$

whose solution is given by

$$Q(n, u_n) = \alpha(n)u_n + \beta(n). \tag{4.4}$$

Next we substitute (4.4) into (4.3), we get

$$\alpha(n+1) - \frac{2}{u_{n+1}}(\alpha(n+1)u_{n+1} + \beta(n+1)) - \alpha(n) = 0,$$

the equation can be split by gathering together all terms with the same dependence upon u_{n+1}

$$-\alpha(n+1) - \alpha(n) - \frac{2}{u_{n+1}}\beta(n+1) = 0.$$

Now, we compare the two sides of the last equation, to obtain

$$-\alpha(n+1) - \alpha(n) = 0,$$

which is a first order linear difference equation whose general solution is

$$\alpha(n) = c(-1)^n,$$

where c is a constant. We have also

$$\beta(n+1) = 0 \quad \text{which implies} \quad \beta(n) = 0.$$

So

$$Q(n, u_n) = (-1)^n u_n.$$

We want to find the invariant using equation (3.24),

$$\frac{du_n}{(-1)^n u_n} = \frac{du_{n+1}}{(-1)^{n+1}u_{n+1}} = \frac{dv_n}{0},$$

Taking the first $\left(\frac{du_n}{(-1)^n u_n}\right)$ and second $\left(\frac{du_{n+1}}{(-1)^{n+1}u_{n+1}}\right)$ invariants, we get

$$\ln|u_n| + c^* = -\ln|u_{n+1}| \quad \text{which implies} \quad -c^* = \ln|u_{n+1}u_n|,$$

where $c^* \in \mathbb{R}$, so

$$k_1 = u_n u_{n+1} \quad \text{where} \quad k_1 = e^{-c^*},$$

also, we have

$$\frac{du_n}{u_n} = \frac{dv_n}{0},$$

which implies that

$$v_n = k, \quad \text{such that} \quad k = f(k_1),$$

where k_1 and k are constants.
We choose $f(k_1) = k_1$, therefore

$$v_n = u_n u_{n+1}. \tag{4.5}$$

Applying the shift operator to v_n yields

$$\begin{aligned} Sv_n = v_{n+1} &= u_{n+1}u_{n+2} \\ &= u_{n+1}\left(\frac{n + u_n u_{n+1}}{u_{n+1}}\right) \\ &= n + u_n u_{n+1} \\ &= n + v_n, \end{aligned} \tag{4.6}$$

So we have the equation

$$v_{n+1} - v_n = n,$$

which is a first order linear difference equation whose solution is given by

$$\begin{aligned} v_n &= v_0 + \sum_{k=0}^{n-1} k \\ &= v_0 + \frac{(n-1)n}{2}. \end{aligned} \tag{4.7}$$

Then by equations (4.5) and (4.7) we have

$$v_n = u_n u_{n+1} = v_0 + \frac{(n-1)n}{2},$$

Solving for u_{n+1} we obtain

$$u_{n+1} = \frac{v_0}{u_n} + \frac{(n-1)n}{2u_n}. \tag{4.8}$$

The order of Equation (4.1) has been reduced by one.
To solve equation (4.8) we need to obtain a canonical coordinate,

$$\begin{aligned} s_n &= \int \frac{du_n}{(-1)^n u_n} \\ &= (-1)^n \ln|u_n|. \end{aligned}$$

So $s_{n+1} - s_n$ is an invariant. Consequently,

$$\begin{aligned} s_{n+1} - s_n &= (-1)^{n+1} \ln|u_{n+1}| - (-1)^n \ln|u_n| \\ &= (-1)^{n+1} \ln|u_n u_{n+1}| \\ &= (-1)^{n+1} \ln|v_n| \\ &= (-1)^{n+1} \ln\left|v_0 + \frac{(n-1)n}{2}\right|, \end{aligned} \tag{4.9}$$

The general solution of (4.9) is

$$\begin{aligned} s_n &= s_0 + \sum_{k=0}^{n-1} (-1)^{k+1} \ln|v_k| \\ &= \ln|u_0| + \sum_{k=0}^{n-1} (-1)^{k+1} \ln\left|u_0 u_1 + \frac{k(k-1)}{2}\right|, \end{aligned}$$

but $s_n = (-1)^n \ln|u_n|$, so the general solution of (4.1)

$$\begin{aligned} u_n &= \exp\left((-1)^n \ln|u_0| + \sum_{k=0}^{n-1} (-1)^{k-n+1} \ln\left|u_0 u_1 + \frac{k(k-1)}{2}\right|\right) \\ &= \exp\left((-1)^n \ln|u_0|\right) \cdot \exp\left(\sum_{k=0}^{n-1} (-1)^{k-n+1} \ln\left|u_0 u_1 + \frac{k(k-1)}{2}\right|\right) \\ &= (u_0)^{(-1)^n} \prod_{k=0}^{n-1} \left(u_0 u_1 + \frac{k(k-1)}{2}\right)^{(-1)^{k-n+1}} \end{aligned}$$

4.2 Exact Solution Of The Difference Equation $u_{n+3} = 1/(u_{n+2}(1+u_n u_{n+1}))$

In this section, we investigate symmetries and solutions of the third-order difference equation $u_{n+3} = 1/(u_{n+2}(1 + u_n u_{n+1}))$.
Consider the third order difference equation

$$u_{n+3} = \frac{1}{u_{n+2}(1 + u_n u_{n+1})}. \tag{4.10}$$

We want to find the solution of equation (4.10) using Lie symmetries.
Firstly, we write the LSC to obtain the characteristics $Q(n, u_n)$,

$$S^3 Q(n, u_n) - \frac{\partial w}{\partial u_n} Q(n, u_n) - \frac{\partial w}{\partial u_{n+1}} Q(n+1, u_{n+1}) - \frac{\partial w}{u_{n+2}} Q(n+2, u_{n+2}) = 0,$$

but

$$\frac{\partial w}{\partial u_n} = \frac{-u_{n+1}}{u_{n+2}(1+u_n u_{n+1})^2} = -u_{n+1}u_{n+2}w^2,$$
$$\frac{\partial w}{\partial u_{n+1}} = \frac{-u_n}{u_{n+2}(1+u_n u_{n+1})^2} = -u_n u_{n+2}w^2,$$

and

$$\frac{\partial w}{\partial u_{n+2}} = \frac{-1}{u_{n+2}^2(1+u_n u_{n+1})} = \frac{-w}{u_{n+2}},$$

so the LSC is

$$Q(n+3,w)+u_{n+1}u_{n+2}w^2Q(n,u_n)+u_n u_{n+2}w^2Q(n+1,u_{n+1})+\frac{w}{u_{n+2}}Q(n+2,u_{n+2}) = 0. \tag{4.11}$$

Now, we apply the differential operator L, given by

$$L = \frac{\partial}{\partial u_n} - \frac{u_{n+1}}{u_n}\frac{\partial}{\partial u_{n+1}},$$

to equation (4.10) to get

$$\frac{\partial}{\partial u_n}\left(Q(n+3,w)+u_{n+1}u_{n+2}w^2Q(n,u_n)+u_n u_{n+2}w^2Q(n+1,u_{n+1})+\frac{w}{u_{n+2}}Q(n+2,u_{n+2})\right)$$
$$-\left(\frac{u_{n+1}}{u_n}\right)\frac{\partial}{\partial u_{n+1}}\Bigg(Q(n+3,w)+u_{n+1}u_{n+2}w^2Q(n,u_n)+u_n u_{n+2}w^2Q(n+1,u_{n+1})$$
$$+\frac{w}{u_{n+2}}Q(n+2,u_{n+2})\Bigg) = 0,$$

but

$$\frac{\partial}{\partial u_n}\left(Q(n+3,w)\right) = 0,$$
$$\frac{\partial}{\partial u_n}\left(u_{n+1}u_{n+2}w^2Q(n,u_n)\right) = u_{n+1}u_{n+2}w^2Q'(n,u_n),$$
$$\frac{\partial}{\partial u_n}\left(u_n u_{n+2}w^2Q(n+1,u_{n+1})\right) = u_{n+2}w^2Q(n+1,u_{n+1}),$$
$$\frac{\partial}{\partial u_n}\left(\frac{w}{u_{n+2}}Q(n+2,w)\right) = 0,$$
$$\frac{\partial}{\partial u_{n+1}}\left(Q(n+3,w)\right) = 0,$$

$$\frac{\partial}{\partial u_{n+1}}\Bigg(u_{n+1}u_{n+2}w^2Q(n,u_n)\Bigg) = u_{n+2}w^2Q(n,u_n),$$
$$\frac{\partial}{\partial u_{n+1}}\Bigg(u_nu_{n+2}w^2Q(n+1,u_{n+1})\Bigg) = u_nu_{n+2}w^2Q'(n+1,u_{n+1}),$$
$$\frac{\partial}{\partial u_{n+1}}\Bigg(\frac{w}{u_{n+2}}Q(n+2,w)\Bigg) = 0,$$

this leads to

$$\Bigg(w^2u_{n+1}u_{n+2}Q'(n,u_n) + w^2u_{n+2}Q(n+1,u_{n+1})\Bigg) - \Bigg(\frac{u_{n+1}}{u_n}\Bigg)\Bigg(w^2u_{n+2}Q(n,u_n) + w^2u_nu_{n+2}Q'(n+1,u_{n+1})\Bigg) = 0,$$

which can be written as

$$w^2u_{n+1}u_{n+2}Q'(n,u_n) + w^2u_{n+2}Q(n+1,u_{n+1}) - \frac{w^2u_{n+1}u_{n+2}}{u_n}Q(n,u_n) - w^2u_{n+1}u_{n+2}Q'(n+1,u_{n+1}) = 0,$$

multiply the last equation by $\frac{1}{u_{n+1}u_{n+2}w^2}$, we obtain

$$Q'(n,u_n) + \frac{1}{u_{n+1}}Q(n+1,u_{n+1}) - \frac{1}{u_n}Q(n,u_n) - Q'(n+1,u_{n+1}) = 0, \tag{4.12}$$

now, we differentiate equation (4.12) with respect to u_n keeping u_{n+1} fixed. As a result we obtain the ODE

$$Q''(n,u_n) + \frac{1}{u_n^2}Q(n,u_n) - \frac{1}{u_n}Q'(n,u_n) = 0,$$

multiply this equation by u_n^2, to get

$$u_n^2Q''(n,u_n) - u_nQ'(n,u_n) + Q(n,u_n) = 0,$$

which is an Euler ordinary differential equation, whose solution is given by

$$Q(n,u_n) = \alpha(n)u_n + \beta(n)u_n\ln u_n, \tag{4.13}$$

for some α and β functions of n.
To find $\alpha(n)$ and $\beta(n)$ we substitute (4.13) into (4.12), we get

$$\alpha(n)+\beta(n)+\beta(n)\ln u_n+\alpha(n+1)+\beta(n+1)\ln u_{n+1}-\alpha(n)-\beta(n)\ln u_n-$$
$$\alpha(n+1)-\beta(n+1)-\beta(n+1)\ln u_{n+1}=0,$$

which implies

$$\beta(n)-\beta(n+1)=0,$$

which is a first order difference equation whose general solution is

$$\beta(n)=c_1; \qquad c_1\in\mathbb{R}.$$

We suppose that $\beta(n)=0$ to simplify computation. Next we substitute

$$Q(n,u_n)=\alpha(n)u_n,$$

into equation (4.11) to obtain

$$\alpha(n+3)w+u_{n+1}u_{n+2}w^2(\alpha(n)u_n)+u_nu_{n+2}w^2(\alpha(n+1)u_{n+1})+\frac{w}{u_{n+2}}(\alpha(n+2)u_{n+2})=0,$$

which implies

$$\Bigg(\alpha(n+3)+\alpha(n+2)\Bigg)w+\Bigg(\alpha(n)+\alpha(n+1)\Bigg)u_nu_{n+1}u_{n+2}w^2=0,$$

to simplify this equation, we substitute $w=\frac{1}{u_{n+2}(1+u_nu_{n+1})}$ and we multiply it by $u_{n+2}(1+u_nu_{n+1})^2$ to obtain

$$\Bigg(\alpha(n+3)+\alpha(n+2)\Bigg)(1+u_nu_{n+1})+\Bigg(\alpha(n)+\alpha(n+1)\Bigg)u_nu_{n+1}=0,$$

this leads to

$$\Bigg(\alpha(n+3)+\alpha(n+2)\Bigg)+\Bigg(\alpha(n+3)+\alpha(n+2)+\alpha(n+1)+\alpha(n)\Bigg)u_nu_{n+1}=0,$$

comparing the two sides, we get

$$\alpha(n+3)+\alpha(n+2)=0,$$

and

$$\alpha(n+3)+\alpha(n+2)+\alpha(n+1)+\alpha(n)=0,$$

Thus,

$$\alpha(n+1) + \alpha(n) = 0,$$

which is a first order linear difference equation whose solution is

$$\alpha(n) = c(-1)^n, \quad \text{where } \ c \ \text{ is a constant.}$$

So:

$$Q(n, u_n) = c(-1)^n u_n, \qquad \text{where } c \text{ is a constant.}$$

Now we want to find the invariant using equation (3.39),

$$\frac{du_n}{(-1)^n u_n} = \frac{du_{n+1}}{(-1)^{n+1} u_{n+1}} = \frac{du_{n+2}}{(-1)^{n+2} u_{n+2}} = \frac{dv_n}{0}.$$

Taking the first $\left(\frac{du_n}{(-1)^n u_n}\right)$ and second $\left(\frac{du_{n+1}}{(-1)^{n+1} u_{n+1}}\right)$ invariants, we get

$$\ln|u_n| + c^* = -\ln|u_{n+1}| \ \text{ which implies } \ -c^* = \ln|u_{n+1} u_n|,$$

where $c^* \in \mathbb{R}$, so

$$k_1 = u_n u_{n+1} \ \text{ where } \ k_1 = e^{-c^*},$$

and taking the first $\left(\frac{du_n}{(-1)^n u_n}\right)$ and third $\left(\frac{du_{n+2}}{(-1)^{n+2} u_{n+2}}\right)$ invariants, we get

$$k_2 = \frac{u_{n+2}}{u_n},$$

and taking the second $\left(\frac{du_{n+1}}{(-1)^{n+1} u_{n+1}}\right)$ and third $\left(\frac{du_{n+2}}{(-1)^{n+2} u_{n+2}}\right)$ invariants, we get

$$k_3 = u_{n+1} u_{n+2},$$

also, we have

$$\frac{du_n}{(-1)^n u_n} = \frac{dv_n}{0},$$

which implies that

$$v_n = k, \ \text{ such that } \ k = f(k_1, k_2, k_3),$$

where k_1, k_2, k_3 and k are constants.
We choose $f(k_1, k_2, k_3) = k_3 = u_{n+1} u_{n+2}$, therefore

$$v_n = u_{n+1} u_{n+2}, \tag{4.14}$$

Applying the shift operator to v_n yields

$$\begin{aligned} Sv_n = v_{n+1} &= u_{n+2}u_{n+3} \\ &= u_{n+2}\frac{1}{u_{n+2}(1+u_nu_{n+1})} \\ &= \frac{1}{1+u_nu_{n+1}} \qquad \text{but} \quad u_nu_{n+1} = v_{n-1} \\ &= \frac{1}{1+v_{n-1}}, \end{aligned}$$

hence

$$v_{n+2} = \frac{1}{1+v_n},$$

which is a second order difference equation that can be solved recursively. Let v_0 and v_1 be given, then

$$v_2 = \frac{1}{1+v_0},$$

$$v_3 = \frac{1}{1+v_1},$$

$$v_4 = \frac{1}{1+v_2} = \frac{1+v_0}{2+v_0},$$

$$v_5 = \frac{1}{1+v_3} = \frac{1+v_1}{2+v_1},$$

$$v_6 = \frac{1}{1+v_4} = \frac{2+v_0}{3+2v_0},$$

$$v_7 = \frac{1}{1+v_5} = \frac{2+v_1}{3+2v_1},$$

$$v_8 = \frac{1}{1+v_6} = \frac{3+2v_0}{5+3v_0},$$

Let $f(n)$ be the Fibonacci numbers which satisfy the recurrence relation

$$f(n) = f(n-1) + f(n-2); \quad n \geq 2,$$

where $f(0) = 0$ and $f(1) = 1$. This is a *second* order linear difference equation whose general solution is given by

$$f(n) = \frac{\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{\sqrt{5}}{5}\left(\frac{1-\sqrt{5}}{2}\right)^n. \tag{4.15}$$

We note that

$$v_2 = \frac{f(1)+f(0)v_0}{f(2)+f(1)v_0},$$

$$v_3 = \frac{f(1)+f(0)v_1}{f(2)+f(1)v_1},$$

$$v_4 = \frac{f(2) + f(1)v_0}{f(3) + f(1)v_0},$$

$$v_5 = \frac{f(2) + f(1)v_1}{f(3) + f(1)v_1},$$

so in general

$$v_n = \begin{cases} \frac{f(\frac{n}{2})+f(\frac{n-2}{2})v_0}{f(\frac{n+2}{2})+f(\frac{n}{2})v_0}; & n = 2k;\ k = 1, 2, 3, \cdots \\ \\ \frac{f(\frac{n-1}{2})+f(\frac{n-3}{2})v_1}{f(\frac{n+1}{2})+f(\frac{n-1}{2})v_1}; & n = 2k+1;\ k = 1, 2, 3, \cdots \end{cases} \tag{4.16}$$

where $f(n)$ is given by (4.15).

Lemma 4.2.1. *The general solution of the difference equation*

$$v_{n+2} = \frac{1}{1 + v_n},$$

is given by (4.16).

Proof. By induction.

Firstly, we want to prove for $n = 2k$; $k = 1, 2, 3, \cdots$

It's true for $k = 1$, (that is $n = 2$) since

$$v_2 = \frac{f(1) + f(0)v_0}{f(2) + f(1)v_0}.$$

Suppose it's true for $k - 1$, (that is $n - 2 = 2k - 2$)

$$v_{n-2} = \frac{f(\frac{n-2}{2}) + f(\frac{n-4}{2})v_0}{f(\frac{n}{2}) + f(\frac{n-2}{2})v_0}.$$

Now, we want to prove for k, (that is $n = 2k$)

$$\begin{aligned} v_n &= \frac{1}{1 + v_{n-2}}, \quad \text{we substitute } v_{n-2} \text{ from our assumption} \\ &= \frac{1}{1 + \frac{f(\frac{n-2}{2})+f(\frac{n-4}{2})v_0}{f(\frac{n}{2})+f(\frac{n-2}{2})v_0}} \\ &= \frac{f(\frac{n}{2}) + f(\frac{n-2}{2})v_0}{f(\frac{n}{2}) + f(\frac{n-2}{2})v_0 + f(\frac{n-2}{2}) + f(\frac{n-4}{2})v_0}, \end{aligned}$$

since $f(n) = f(n-1) + f(n-2)$, we get

$$v_n = \frac{f(\frac{n}{2}) + f(\frac{n-2}{2})v_0}{f(\frac{n+2}{2}) + f(\frac{n}{2})v_0}.$$

Secondly, we want to prove it if $n = 2k+1$; $k = 1, 2, 3, \cdots$
It's true for $k = 1$, (that is $n = 3$) since

$$v_3 = \frac{f(1) + f(0)v_1}{f(2) + f(1)v_1}.$$

Suppose it's true for $k-1$, that is $n-2 = 2k-1$

$$v_{n-2} = \frac{f(\frac{n-3}{2}) + f(\frac{n-5}{2})v_1}{f(\frac{n-1}{2}) + f(\frac{n-3}{2})v_1}.$$

Now, we want to prove for k, (that is $n = 2k+1$)

$$\begin{aligned} v_n &= \frac{1}{1+v_{n-2}}, \quad \text{we substitute } v_{n-2} \text{ from our assumption} \\ &= \frac{1}{1 + \frac{f(\frac{n-3}{2})+f(\frac{n-5}{2})v_1}{f(\frac{n-1}{2})+f(\frac{n-3}{2})v_1}} \\ &= \frac{f(\frac{n-1}{2}) + f(\frac{n-3}{2})v_1}{f(\frac{n-1}{2}) + f(\frac{n-3}{2})v_1 + f(\frac{n-3}{2}) + f(\frac{n-5}{2})v_1}, \end{aligned}$$

since $f(n) = f(n-1) + f(n-2)$, we get

$$v_n = \frac{f(\frac{n-1}{2}) + f(\frac{n-3}{2})v_1}{f(\frac{n+1}{2}) + f(\frac{n-1}{2})v_1}.$$

□

This proves our result.

Then using equation (4.14) and equation (4.16) and solving for u_{n+2} we obtain

$$u_{n+2} = \frac{v_n}{u_{n+1}}, \tag{4.17}$$

where v_n is given by equation (4.16). The order of equation (4.10) has been reduced by two.
To solve equation (4.17) we need to obtain a canonical coordinate s_n,

$$\begin{aligned} s_n = &\int \frac{du_n}{(-1)^n u_n} \\ =&(-1)^n \ln \mid u_n \mid . \end{aligned}$$

So $s_{n+1} - s_n$ is an invariant. Consequently,

$$\begin{aligned} s_{n+1} - s_n &= (-1)^{n+1} \ln|u_{n+1}| - (-1)^n \ln|u_n| \\ &= (-1)^{n+1} \ln|u_n u_{n+1}|, \end{aligned} \tag{4.18}$$

which is a first order difference equation whose general solution is

$$\begin{aligned} s_n &= s_0 + \sum_{k=0}^{n-1} (-1)^{k+1} \ln|u_k u_{k+1}| \\ &= \ln|u_0| + (-1)^1 \ln|u_0 u_1| + (-1)^2 \ln|u_1 u_2| + (-1)^3 \ln|u_2 u_3| + \sum_{k=3}^{n-1} (-1)^{k+1} \ln|v_{k-1}| \\ &= -\ln|u_3| + \sum_{k=3}^{n-1} (-1)^{k+1} \ln|v_{k-1}|, \end{aligned} \tag{4.19}$$

where $u_3 = \frac{1}{u_2(1+u_0 u_1)}$ and v_{k-1} is given by

$$v_{k-1} = \begin{cases} \frac{f(\frac{k-1}{2})+f(\frac{k-3}{2})v_0}{f(\frac{k+1}{2})+f(\frac{k-1}{2})v_0}; & k = 3, 5, 7, \cdots \\ \\ \frac{f(\frac{k-2}{2})+f(\frac{k-4}{2})v_1}{f(\frac{k}{2})+f(\frac{k-2}{2})v_1}; & k = 4, 6, 8, \cdots \end{cases} \tag{4.20}$$

where $v_0 = u_1 u_2$, $v_1 = u_2 u_3 = \frac{1}{1+u_0 u_1}$ and f is given by (4.15) .
Also, we have $s_n = (-1)^n \ln|u_n|$, so

$$u_n = \exp\left((-1)^n s_n\right), \tag{4.21}$$

Now, from equation (4.19) and equation (4.21), we obtain the general solution to equation (4.10)

$$\begin{aligned} u_n &= \exp\left((-1)^n(-\ln|u_3| + \sum_{k=3}^{n-1} (-1)^{k+1} \ln|v_{k-1}|)\right) \\ &= \exp\left((-1)^{n+1} \ln\left|\frac{1}{u_2(1+u_0 u_1)}\right| + \sum_{k=3}^{n-1} (-1)^{k+n+1} \ln|v_{k-1}|\right), \end{aligned} \tag{4.22}$$

where u_0, u_1, and u_2 are given and v_{k-1} is given by equation (4.20).
To verify that equation (4.22) solves equation (4.10)

$$\begin{aligned} u_n &= \exp\left((-1)^{n+1} \ln\left|\frac{1}{u_2(1+u_0 u_1)}\right| + \sum_{k=3}^{n-1} (-1)^{k+n+1} \ln|v_{k-1}|\right) \\ &= \exp\left((-1)^{n+1} \ln\left|\frac{1}{u_2(1+u_0 u_1)}\right|\right) \exp\left(\sum_{k=3}^{n-1} (-1)^{k+n+1} \ln|v_{k-1}|\right) \\ &= \left(\frac{1}{u_2(1+u_0 u_1)}\right)^{(-1)^{n+1}} \left(\prod_{k=3}^{n-1} (v_{k-1})^{(-1)^{k+n+1}}\right) \\ &= \left(\left[u_2(1+u_0 u_1)\right]^{-1}\right)^{(-1)^{n+1}} \left(\prod_{k=3}^{n-1} (v_{k-1})^{(-1)^{k+n+1}}\right) \end{aligned}$$

$$= \left[u_2(1+u_0u_1)\right]^{(-1)^{n+2}} \left(\prod_{k=3}^{n-1}(v_{k-1})^{(-1)^{k+n+1}}\right)$$
$$= \left[u_2(1+u_0u_1)\right]^{(-1)^{n}} \left(\prod_{k=3}^{n-1}(v_{k-1})^{(-1)^{k+n+1}}\right)$$

and

$$u_{n+1} = \left[u_2(1+u_0u_1)\right]^{(-1)^{n+1}} \left(\prod_{k=3}^{n}(v_{k-1})^{(-1)^{k+n}}\right)$$
$$u_{n+2} = \left[u_2(1+u_0u_1)\right]^{(-1)^{n}} \left(\prod_{k=3}^{n+1}(v_{k-1})^{(-1)^{k+n+1}}\right)$$
$$u_{n+3} = \left[u_2(1+u_0u_1)\right]^{(-1)^{n+1}} \left(\prod_{k=3}^{n+2}(v_{k-1})^{(-1)^{k+n}}\right)$$

now, from this we have

$$\begin{aligned}
u_nu_{n+1} &= \left[u_2(1+u_0u_1)\right]^{(-1)^{n}} \left(\prod_{k=3}^{n-1}(v_{k-1})^{(-1)^{k+n+1}}\right)\left[u_2(1+u_0u_1)\right]^{(-1)^{n+1}} \left(\prod_{k=3}^{n}(v_{k-1})^{(-1)^{k+n}}\right) \\
&= \left[u_2(1+u_0u_1)\right]^{(-1)^{n}-(-1)^{n}} \left(\prod_{k=3}^{n-1}(v_{k-1})^{(-1)^{k+n+1}}(v_{k-1})^{(-1)^{k+n}}\right)\left(v_{n-1}\right)^{(-1)^{2n}} \\
&= (1)\left(\prod_{k=3}^{n-1}(v_{k-1})^{(-1)^{k+n+1}+(-1)^{k+n}}\right)\left(v_{n-1}\right) \\
&= \left(\prod_{k=3}^{n-1}(v_{k-1})^{0}\right)\left(v_{n-1}\right) \\
&= v_{n-1},
\end{aligned}$$

so,

$$u_{n+2}(1+u_nu_{n+1}) = \left[u_2(1+u_0u_1)\right]^{(-1)^{n}} \left(\prod_{k=3}^{n+1}(v_{k-1})^{(-1)^{k+n+1}}\right)\left(1+v_{n-1}\right),$$

from this we get

$$\begin{aligned}
\frac{1}{u_{n+2}(1+u_nu_{n+1})} &= \frac{1}{\left[u_2(1+u_0u_1)\right]^{(-1)^{n}} \left(\prod_{k=3}^{n+1}(v_{k-1})^{(-1)^{k+n+1}}\right)\left(1+v_{n-1}\right)} \\
&= \left[u_2(1+u_0u_1)\right]^{(-1)^{n+1}} \left(\frac{1}{\prod_{k=3}^{n+1}(v_{k-1})^{(-1)^{k+n+1}}}\right)\left(\frac{1}{1+v_{n-1}}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \left[u_2(1+u_0u_1)\right]^{(-1)^{n+1}} \left(\prod_{k=3}^{n+1} (\frac{1}{v_{k-1}})^{(-1)^{k+n+1}}\right)\left(v_{n+1}\right)\\
&= \left[u_2(1+u_0u_1)\right]^{(-1)^{n+1}} \left(\prod_{k=3}^{n+1} (v_{k-1})^{(-1)^{k+n+2}}\right)\left(v_{n+1}\right)\\
&= \left[u_2(1+u_0u_1)\right]^{(-1)^{n+1}} \left(\prod_{k=3}^{n+1} (v_{k-1})^{(-1)^{k+n}}\right)\left(v_{n+1}\right)\\
&= \left[u_2(1+u_0u_1)\right]^{(-1)^{n+1}} \left(\prod_{k=3}^{n+2} (v_{k-1})^{(-1)^{k+n}}\right)\\
&= u_{n+3}.
\end{aligned}
$$

This proves that equation (4.22) is a solution of the equation (4.10).

4.3 Symmetry Analysis And Exact Solution Of The Difference Equation $u_{n+4} = (u_n u_{n+1})/(u_n + u_{n+3})$

In this section, we investigate the solution of the fourth order difference equation $u_{n+4} = (u_n u_{n+1})/(u_n + u_{n+3})$ using Lie symmetries.
Consider the fourth order difference equation

$$
u_{n+4} = \frac{u_n u_{n+1}}{u_n + u_{n+3}}. \tag{4.23}
$$

To find the general solution using Lie symmetries, we write the *LSC*

$$
Q(n+4, u_{n+4}) - \frac{\partial w}{\partial u_n} Q(n, u_n) - \frac{\partial w}{\partial u_{n+1}} Q(n+1, u_{n+1}) - \frac{\partial w}{\partial u_{n+2}} Q(n+2, u_{n+2}) - \frac{\partial w}{\partial u_{n+3}} Q(n+3, u_{n+3}) = 0,
$$

but

$$
\frac{\partial w}{\partial u_n} = \frac{u_{n+1}u_{n+3}}{(u_n + u_{n+3})^2} = \frac{w^2 u_{n+3}}{u_n^2 u_{n+1}},
$$

$$
\frac{\partial w}{\partial u_{n+1}} = \frac{u_n}{u_n + u_{n+3}} = \frac{w}{u_{n+1}},
$$

$$
\frac{\partial w}{\partial u_{n+2}} = 0,
$$

and

$$
\frac{\partial w}{\partial u_{n+3}} = \frac{-u_n u_{n+1}}{(u_n + u_{n+3})^2} = \frac{-w^2}{u_n u_{n+1}},
$$

so the LSC is

$$Q(n+4,w)-\frac{w^2u_{n+3}}{u_n^2u_{n+1}}Q(n,u_n)-\frac{w}{u_{n+1}}Q(n+1,u_{n+1})+\frac{w^2}{u_nu_{n+1}}Q(n+3,u_{n+3})=0. \quad (4.24)$$

Now, we apply the differential operator (L), given by

$$\begin{aligned} L &= \frac{\partial}{\partial u_n}+\frac{\partial u_{n+1}}{\partial u_n}\frac{\partial}{\partial u_{n+1}} \\ &= \frac{\partial}{\partial u_n}-\frac{wu_{n+3}}{u_n^2}\frac{\partial}{\partial u_{n+1}}, \end{aligned}$$

to get

$$\frac{\partial}{\partial u_n}\left(Q(n+4,w)-\frac{w^2u_{n+3}}{u_n^2u_{n+1}}Q(n,u_n)-\frac{w}{u_{n+1}}Q(n+1,u_{n+1})+\frac{w^2}{u_nu_{n+1}}Q(n+3,u_{n+3})\right)-$$
$$\left(\frac{wu_{n+3}}{u_n^2}\frac{\partial}{\partial u_{n+1}}\right)\left(Q(n+4,w)-\frac{w^2u_{n+3}}{u_n^2u_{n+1}}Q(n,u_n)-\frac{w}{u_{n+1}}Q(n+1,u_{n+1})+\right.$$
$$\left.\frac{w^2}{u_nu_{n+1}}Q(n+3,u_{n+3})\right)=0,$$

but

$$\begin{aligned}
&\frac{\partial}{\partial u_n}\left(Q(n+4,u_{n+4})\right)=0,\\
&\frac{\partial}{\partial u_n}\left(\frac{w^2u_{n+3}}{u_n^2u_{n+1}}Q(n,u_n)\right)=\frac{w^2u_{n+3}}{u_n^2u_{n+1}}Q'(n,u_n)-\frac{2w^2u_{n+3}}{u_n^3u_{n+1}}Q(n,u_n),\\
&\frac{\partial}{\partial u_n}\left(\frac{w}{u_{n+1}}Q(n+1,u_{n+1})\right)=0,\\
&\frac{\partial}{\partial u_n}\left(\frac{w^2}{u_nu_{n+1}}Q(n+3,u_{n+3})\right)=\frac{-w^2}{u_n^2u_{n+1}}Q(n+3,u_{n+3}),\\
&\frac{\partial}{\partial u_{n+1}}\left(Q(n+4,u_{n+4})\right)=0,\\
&\frac{\partial}{\partial u_{n+1}}\left(\frac{w^2u_{n+3}}{u_n^2u_{n+1}}Q(n,u_n)\right)=\frac{-w^2u_{n+3}}{u_n^2u_{n+1}^2}Q(n,u_n),\\
&\frac{\partial}{\partial u_{n+1}}\left(\frac{w}{u_{n+1}}Q(n+1,u_{n+1})\right)=\frac{w}{u_{n+1}}Q'(n+1,u_{n+1})+\frac{-w}{u_{n+1}^2}Q(n+1,u_{n+1}),\\
&\frac{\partial}{\partial u_{n+1}}\left(\frac{w^2}{u_nu_{n+1}}Q(n+3,u_{n+3})\right)=\frac{-w^2}{u_nu_{n+1}^2}Q(n+3,u_{n+3}),
\end{aligned}$$

so

$$\frac{-w^2u_{n+3}}{u_n^2u_{n+1}}Q'(n,u_n)+\frac{2w^2u_{n+3}}{u_n^3u_{n+1}}Q(n,u_n)+\frac{-w^2}{u_n^2u_{n+1}}Q(n+3,u_{n+3})$$
$$-\left(\frac{wu_{n+3}}{u_n^2}\right)\left[\frac{w^2u_{n+3}}{u_n^2u_{n+1}^2}Q(n,u_n)-\frac{w}{u_{n+1}}Q'(n+1,u_{n+1})+\frac{w}{u_{n+1}^2}Q(n+1,u_{n+1})-\right.$$
$$\left.\frac{w^2}{u_nu_{n+1}^2}Q(n+3,u_{n+3})\right]=0,$$

this leads to

$$\frac{-w^2u_{n+3}}{u_n^2u_{n+1}}Q'(n,u_n)+\frac{2w^2u_{n+3}}{u_n^3u_{n+1}}Q(n,u_n)-\frac{w^2}{u_n^2u_{n+1}}Q(n+3,u_{n+3})-$$
$$\frac{w^3u_{n+3}^2}{u_n^4u_{n+1}^2}Q(n,u_n)+\frac{w^2u_{n+3}}{u_n^2u_{n+1}}Q'(n+1,u_{n+1})-\frac{w^2u_{n+3}}{u_n^2u_{n+1}^2}Q(n+1,u_{n+1})+$$
$$\frac{w^3u_{n+3}}{u_n^3u_{n+1}^2}Q(n+3,u_{n+3})=0,$$

multiply the last equation by $\frac{-u_n^2u_{n+1}}{w^2u_{n+3}}$, to get

$$Q'(n,u_n)-\frac{2}{u_n}Q(n,u_n)+\frac{1}{u_{n+3}}Q(n+3,u_{n+3})+\frac{wu_{n+3}}{u_n^2u_{n+1}}Q(n,u_n)$$
$$-Q'(n+1,u_{n+1})+\frac{1}{u_{n+1}}Q(n+1,u_{n+1})-\frac{w}{u_nu_{n+1}}Q(n+3,u_{n+3})=0. \quad (4.25)$$

Now, differentiate equation (4.25) with respect to u_n keeping u_{n+1} fixed

$$Q''(n,u_n)-\frac{2}{u_n}Q'(n,u_n)+\frac{2}{u_n^2}Q(n,u_n)+\frac{wu_{n+3}}{u_n^2u_{n+1}}Q'(n,u_n)-\frac{2wu_{n+3}}{u_n^3u_{n+1}}Q(n,u_n)+$$
$$\frac{w}{u_n^2u_{n+1}}Q(n+3,u_{n+3})=0,$$

multiply this equation by u_n^2, to get

$$u_n^2Q''(n,u_n)-2u_nQ'(n,u_n)+2Q(n,u_n)+\frac{wu_{n+3}}{u_{n+1}}Q'(n,u_n)-\frac{2wu_{n+3}}{u_nu_{n+1}}Q(n,u_n)+$$
$$\frac{w}{u_{n+1}}Q(n+3,u_{n+3})=0, \quad (4.26)$$

again, we differentiate with respect to u_n

$$u_n^2Q'''(n,u_n)+\frac{wu_{n+3}}{u_{n+1}}Q''(n,u_n)-\frac{2wu_{n+3}}{u_nu_{n+1}}Q'(n,u_n)+\frac{2wu_{n+3}}{u_n^2u_{n+1}}Q(n,u_n)=0.$$

To simplify this differential equation we substitute $w = \frac{u_n u_{n+1}}{u_n+u_{n+3}}$, then multiply by $u_n(u_n + u_{n+3})$, to obtain

$$u_n^3(u_n + u_{n+3})Q'''(n, u_n) + u_n^2 u_{n+3}Q''(n, u_n) - 2u_n u_{n+3}Q'(n, u_n) + 2u_{n+3}Q(n, u_n) = 0,$$

which implies

$$u_n^4 Q'''(n, u_n) + u_n^3 u_{n+3}Q'''(n, u_n) + u_n^2 u_{n+3}Q''(n, u_n) - 2u_n u_{n+3}Q'(n, u_n)+ 2u_{n+3}Q(n, u_n) = 0,$$

since $Q(n, u_n)$ depends on n and u_n only, we separate the last equation with respect to u_{n+3}.
The coefficient of 1 is

$$u_n^4 Q'''(n, u_n) = 0, \tag{4.27}$$

and the coefficient of u_{n+3} is

$$u_n^3 Q'''(n, u_n) + u_n^2 Q''(n, u_n) - 2u_n Q'(n, u_n) + 2Q(n, u_n) = 0. \tag{4.28}$$

from equation (4.27) we get $Q'''(n, u_n) = 0$, so equation (4.28) becomes

$$u_n^2 Q''(n, u_n) - 2u_n Q'(n, u_n) + 2Q(n, u_n) = 0,$$

which is a Cauchy Euler differential equation whose solution is given by

$$Q(n, u_n) = \alpha(n)u_n + \beta(n)u_n^2, \tag{4.29}$$

where $\alpha(n)$ and $\beta(n)$ are functions of n.
Next we substitute (4.29) into (4.26), we get

$$u_n^2(2\beta(n)) - 2u_n(\alpha(n)+2\beta(n)u_n)+2\alpha(n)u_n+2\beta(n)u_n^2+\frac{wu_{n+3}}{u_{n+1}}\alpha(n)+\frac{2wu_{n+3}}{u_{n+1}}(\beta(n)u_n) -\frac{2wu_{n+3}}{u_n u_{n+1}}(\alpha(n)u_n)-\frac{2wu_{n+3}}{u_n u_{n+1}}(\beta(n)u_n^2)+\frac{w}{u_{n+1}}(\alpha(n+3)u_{n+3})+\frac{w}{u_{n+1}}(\beta(n+3)u_{n+3}^2) = 0,$$

this leads to

$$\frac{wu_{n+3}}{u_{n+1}}(\alpha(n) - 2\alpha(n) + \alpha(n + 3)) + \frac{wu_{n+3}^2}{u_{n+1}}(\beta(n + 3)) = 0,$$

multiply by $\frac{u_{n+1}}{wu_{n+3}}$, we get

$$(\alpha(n + 3) - \alpha(n)) + \beta(n + 3)u_{n+3} = 0,$$

comparing the two sides of the last equation, we get

$$\alpha(n+3) - \alpha(n) = 0,$$

which is a third order linear difference equation whose solution is given by

$$\begin{aligned}\alpha(n) &= c_1 + c_2\left(\frac{-1+\sqrt{3}i}{2}\right)^n + c_3\left(\frac{-1-\sqrt{3}i}{2}\right)^n \\ &= c_1 + c_2\left(\cos(\frac{2n\pi}{3}) + i\sin(\frac{2n\pi}{3})\right) + c_3\left(\cos(\frac{2n\pi}{3}) - i\sin(\frac{2n\pi}{3})\right),\end{aligned}$$

where c_1, c_2 and $c_3 \in \mathbb{R}$. and

$$\beta(n+3) = 0 \quad \text{so} \quad \beta(n) = 0, \quad \text{for all } n.$$

Hence,

$$Q(n, u_n) = \left[c_1 + c_2\left(\cos(\frac{2n\pi}{3}) + i\sin(\frac{2n\pi}{3})\right) + c_3\left(\cos(\frac{2n\pi}{3}) - i\sin(\frac{2n\pi}{3})\right)\right]u_n.$$

We suppose that $c_2 = 0$ and $c_3 = 0$ to simplify computation. So

$$Q(n, u_n) = c_1 u_n, \text{ where } \ c_1 \ \text{ is a constant.}$$

Now, we want to find the invariant using equation (3.39),

$$\frac{du_n}{u_n} = \frac{du_{n+1}}{u_{n+1}} = \frac{du_{n+2}}{u_{n+2}} = \frac{du_{n+3}}{u_{n+3}} = \frac{dv_n}{0}.$$

Taking the first $(\frac{du_n}{u_n})$ and second $(\frac{du_{n+1}}{u_{n+1}})$ invariants, we get

$$\ln u_n + c^* = \ln u_{n+1} \ \text{ which implies } \ c^* = \ln\frac{u_{n+1}}{u_n},$$

where $c^* \in \mathbb{R}$, so

$$k_1 = \frac{u_{n+1}}{u_n}, \quad \text{where} \quad k_1 = e^{c^*},$$

taking the first $(\frac{du_n}{u_n})$ and third $(\frac{du_{n+2}}{u_{n+2}})$ invariants, we get

$$k_2 = \frac{u_{n+2}}{u_n}, \quad \text{where} \quad k_2 \in \mathbb{R}$$

taking the first $(\frac{du_n}{u_n})$ and fourth $(\frac{du_{n+3}}{u_{n+3}})$ invariants, we get

$$k_3 = \frac{u_{n+3}}{u_n}, \quad \text{where} \quad k_3 \in \mathbb{R}$$

taking the second ($\frac{du_{n+1}}{u_{n+1}}$) and third ($\frac{du_{n+2}}{u_{n+2}}$) invariants, we get

$$k_4 = \frac{u_{n+2}}{u_{n+1}}, \quad \text{where} \quad k_4 \ \in \ \mathbb{R}$$

taking the second ($\frac{du_{n+1}}{u_{n+1}}$) and fourth ($\frac{du_{n+3}}{u_{n+3}}$) invariants, we get

$$k_5 = \frac{u_{n+3}}{u_{n+1}}, \quad \text{where} \quad k_5 \ \in \ \mathbb{R}$$

and taking the third ($\frac{du_{n+2}}{u_{n+2}}$) and fourth ($\frac{du_{n+3}}{u_{n+3}}$) invariants, we get

$$k_6 = \frac{u_{n+3}}{u_{n+2}}, \quad \text{where} \quad k_6 \ \in \ \mathbb{R}$$

also, we have

$$\frac{du_n}{u_n} = \frac{dv_n}{0},$$

which implies that

$$v_n = k, \quad \text{where} \quad k = f(k_1, k_2, k_3, k_4, k_5, k_6),$$

where k_1, k_2, k_3, k_4, k_5, k_6 and k are constants.

We choose $f(k_1, k_2, k_3, k_4, k_5, k_6) = k_3$, therefore

$$v_n = \frac{u_{n+3}}{u_n}, \tag{4.30}$$

Applying the shift operator to v_n yields

$$\begin{aligned} Sv_n = v_{n+1} &= \frac{u_{n+4}}{u_{n+1}} \\ &= \frac{u_n u_{n+1}}{u_{n+1}(u_n + u_{n+3})} \\ &= \frac{u_n}{u_n + u_{n+3}} \\ &= \frac{1}{1 + \frac{u_{n+3}}{u_n}}, \quad \text{but } \frac{u_{n+3}}{u_n} = v_n \\ &= \frac{1}{1 + v_n}. \end{aligned}$$

So we have the equation

$$v_{n+1} = \frac{1}{v_n + 1},$$

which is a Riccati difference equation of type one, where $a(n) = 1$, $b(n) = 0$ and $g(n) = 1$ so to solve it we let

$$v_n = \frac{x_{n+1}}{x_n} - 1,$$

to get

$$x_{n+2} - x_{n+1} - x_n = 0,$$

so

$$x_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n,$$

where c_1, $c_2 \in \mathbb{R}$. this implies

$$\begin{aligned}
v_n &= \frac{x_{n+1}}{x_n} - 1\\
&= \frac{c_1\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} + c_2\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{c_1\left(\frac{1+\sqrt{5}}{2}\right)^{n} + c_2\left(\frac{1-\sqrt{5}}{2}\right)^{n}} - 1\\
&= \frac{2c_1(1+\sqrt{5})^{n-1} + 2c_2(1-\sqrt{5})^{n-1}}{c_1(1+\sqrt{5})^n + c_2(1-\sqrt{5})^n}. \qquad (4.31)
\end{aligned}$$

Then by equations (4.30) and (4.31) we have

$$v_n = \frac{u_{n+3}}{u_n} = \frac{2c_1(1+\sqrt{5})^{n-1} + 2c_2(1-\sqrt{5})^{n-1}}{c_1(1+\sqrt{5})^n + c_2(1-\sqrt{5})^n},$$

solving for u_{n+3} we obtain

$$u_{n+3} = \left(\frac{2c_1(1+\sqrt{5})^{n-1} + 2c_2(1-\sqrt{5})^{n-1}}{c_1(1+\sqrt{5})^n + c_2(1-\sqrt{5})^n}\right)u_n. \qquad (4.32)$$

The order of Equation (4.23) has been reduced by one.

To solve equation (4.32) we need to obtain a canonical coordinate,

$$\begin{aligned}
s_n &= \int \frac{du_n}{u_n}\\
&= \ln \mid u_n \mid .
\end{aligned}$$

So $s_{n+3} - s_n$ is an invariant. Consequently,

$$\begin{aligned}
s_{n+3} - s_n &= \ln|u_{n+3}| - \ln|u_n|\\
&= \ln\left|\frac{u_{n+3}}{u_n}\right|\\
&= \ln|v_n|\\
&= \ln\left|\left(\frac{2c_1(1+\sqrt{5})^{n-1} + 2c_2(1-\sqrt{5})^{n-1}}{c_1(1+\sqrt{5})^n + c_2(1-\sqrt{5})^n}\right)\right|, \qquad (4.33)
\end{aligned}$$

The general solution of (4.33) is

$$\begin{aligned}
s_n &= a_1 + a_2\left(\frac{-1+\sqrt{3}i}{2}\right)^n + a_3\left(\frac{-1-\sqrt{3}i}{2}\right)^n + \\
&\sum_{k=0}^{n-1}\ln\left|\left(\frac{2c_1(1+\sqrt{5})^{k-1}+2c_2(1-\sqrt{5})^{k-1}}{c_1(1+\sqrt{5})^k+c_2(1-\sqrt{5})^k}\right)\right| \\
&= a_1 + a_2\left(\cos(\frac{2n\pi}{3}) + i\sin(\frac{2n\pi}{3})\right) + a_3\left(\cos(\frac{2n\pi}{3}) - i\sin(\frac{2n\pi}{3})\right) \\
&+\sum_{k=0}^{n-1}\ln\left|\left(\frac{2c_1(1+\sqrt{5})^{k-1}+2c_2(1-\sqrt{5})^{k-1}}{c_1(1+\sqrt{5})^k+c_2(1-\sqrt{5})^k}\right)\right| \\
&= a_1 + (a_2+a_3)\cos(\frac{2n\pi}{3}) + i(a_2-a_3)\sin(\frac{2n\pi}{3}) \\
&+\sum_{k=0}^{n-1}\ln\left|\left(\frac{2c_1(1+\sqrt{5})^{k-1}+2c_2(1-\sqrt{5})^{k-1}}{c_1(1+\sqrt{5})^k+c_2(1-\sqrt{5})^k}\right)\right| \\
&= a_1 + a_2'\cos(\frac{2n\pi}{3}) + a_3'\sin(\frac{2n\pi}{3}) + \sum_{k=0}^{n-1}\ln\left|\left(\frac{2c_1(1+\sqrt{5})^{k-1}+2c_2(1-\sqrt{5})^{k-1}}{c_1(1+\sqrt{5})^k+c_2(1-\sqrt{5})^k}\right)\right|,
\end{aligned}$$

where $a_2' = a_2 + a_3$ and $a_3' = i(a_2 - a_3)$.

The canonical coordinate $s_n = \ln|u_n|$, so the general solution of (4.23) is

$$u_n = \exp\left[a_1 + a_2'\cos(\frac{2n\pi}{3}) + a_3'\sin(\frac{2n\pi}{3}) + \sum_{k=0}^{n-1}\ln\left|\left(\frac{2c_1(1+\sqrt{5})^{k-1}+2c_2(1-\sqrt{5})^{k-1}}{c_1(1+\sqrt{5})^k+c_2(1-\sqrt{5})^k}\right)\right|\right].$$

Hence,

$$u_n = \prod_{k=0}^{n-1}\left(\frac{2c_1(1+\sqrt{5})^{k-1}+2c_2(1-\sqrt{5})^{k-1}}{c_1(1+\sqrt{5})^k+c_2(1-\sqrt{5})^k}\right)\cdot\exp\left[a_1+a_2'\cos(\frac{2n\pi}{3})+a_3'\sin(\frac{2n\pi}{3})\right].$$

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